

Convergence of Coloring Games with Collusions

and why are your Facebook “friends” not optimal?

Guillaume Ducoffe
Columbia University
Computer Science Dept
New York, NY, USA
gducoffe@ens-cachan.fr

Dorian Mazauric
Columbia University
Computer Science Dept
New York, NY, USA
doric@cs.columbia.edu

Augustin Chaintreau
Columbia University
Computer Science Dept
New York, NY, USA
augustin@cs.columbia.edu

ABSTRACT

Various social networks, and some computational problems, require forming groups in the presence of *de facto* antagonistic relationships denoting local incompatibility. This can be posed as a vertex graph coloring problem, and the configurations attainable under these constraints through coordinated algorithms (*e.g.*, aiming at using the least amount of colors) are well characterized. In contrast, when groups are formed in a distributed manner by self-interested nodes, we know surprisingly little of the configurations that emerge from the dynamics of such a *coloring game*. Collusions – multiple players joining a new group together so that they *all* improve their scores – are likely to occur in such group formation settings; they greatly affect the stability and efficiency of the configurations.

This paper characterizes the convergence of coloring games, revealing their intricate complexity properties and proving that previous bounds are arbitrarily loose. First, in a uniform game, we show that sequences of moves relate to physics models of condensed matter and integer partition. Moreover, we prove that combinatorial complexity of coloring games suddenly increases: no polynomial bound exists whenever this analysis fails. This solves an open problem and explains the failure of previous proposed methods using potential functions. In non-uniform games where nodes preferences are given in a set of weights, we prove generally the maximum collusion size that guarantees convergence. As we show, deciding the convergence beyond this point is never computationally feasible. Our results extend to variants where nodes join multiple groups, have asymmetrical weights, and utility goes beyond pairwise relationship. Finally, we show these games create a tension between stability and efficiency, as collusions and using multiple colors both have a beneficial effect on the price of anarchy.

1. INTRODUCTION

In many situations, nodes belong to a network that also contains various conflicts and enmity among its members,

or at least a diversity of opinions. In that case, we often are required to *take sides* and form subsets or groups to interact with each other. Formal disagreement among nodes over well-defined societal issues (*e.g.*, “Which person should lead the executive branch of the government for the next 4 years?”) are typically resolved through democratic decision process. But disagreement in many networks most frequently occurs for less formal topics (*e.g.*, “What constitutes a *fun* evening?”, “Which video clip available online is *worth* the time to watch it?”). Moreover, two persons’ views may differ so radically that they are essentially incompatible. In retrospect, this situation is so common that it generally affects how informal groups of social interaction are formed. The formation of social gatherings (*e.g.*, choosing your Facebook groups, or choosing which communities to actively participate in), is primarily driven by the benefit gained from interacting with each person in these groups. Groups are formed in a distributed manner as outsiders become attracted to some of the group’s members. Members may potentially revise their decision and *vote with their feet*, as other groups become more attractive to them.

Understanding how groups form in a social network is similar to a graph coloring problem. In its simplest version, colors should be assigned to vertices of a graph so that edges, which represent conflicts, never connect two people in a same-color group. What is unique in our case is that, instead of nodes following a coordinated decision towards a global objective (*e.g.*, minimizing the number of distinct groups or colors), individuals more generally make *local* and *selfish* decisions, typically *at arbitrary time*. In essence, social group formation is a coloring game with complex distributed dynamics, and this is the perspective we adopt in this paper. In particular we account address three aspects of this problem:

- **Players preferences are arbitrary and private.** We can assume that nodes decide to join groups according to a weighted graph that represents their interests in other nodes. However, it is neither easy nor desirable to disclose these weights. This is why we aim to prove the convergence of *all* games under *any* sequence of local decision.
- **Networks include conflict edges.** The game is hence played over a graph with edges denoting “positive” and “negative” interaction. As an extreme case, one node may wish to avoid another *at all cost*. We include all these cases.
- **Players participate in collusions.** It is frequently

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. To copy otherwise, to republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee.

Copyright 20XX ACM X-XXXXX-XX-X/XX/XX ...\$15.00.

assumed in game theory that nodes can unilaterally deviate but cannot act collectively to increase their return. Here, we wish to relax this assumption, since we deal with how people form groups. Our model details the impact of such collusions on the game’s convergence.

Coloring game: Definition, Examples, Properties.

We define a *coloring game* on a general weighted graph as follows: each edge denotes a potential connection or alliance, and its weight represents how advantageous this connection is to both nodes¹. Edges with positive weights denote desirable connections, while an edge with negative weight, *a fortiori* if the weight is very large, denotes a conflictual relationship. A player can strategically decide which color she uses, knowing that she will then receive a utility which depends on the weights of her neighboring edges towards other players picking the same color². One possible choice is to pick a color no-one is currently using, in which case she will be isolated and by convention will receive a null utility. In addition, a subset of players can decide to act collectively to form a *k-deviation* (where *k* is the size of this subset), by simultaneously changing their colors, as long as *each* of them strictly increases her utility.

Coloring games naturally arise in situations where nodes have positive and negative relationships, and interact after forming various factions:

- In the case of an informal social gathering, one can decide to join an online group or community for the sole purpose of being able to interact with its members, for example to exchange information. However, conflicts are present in the form of persons one wishes to avoid, *e.g.*, an ex-partner, an employer; information disclosure might in that case hurt each others’ privacy.
- In Spread Spectrum telecommunications such as Code Division Multiple Access (CDMA) wireless networks, a group of nodes can decide on a coding scheme to communicate among each other without nodes from other groups either interfering or being able to eavesdrop or even easily detect their communication.
- Vendors for multiple goods could consider *bundling*, which means coupling some products together (*e.g.*, via providing additional services such as a loyalty program, or co-locating shops near to each other in the same mall). Entering an alliance involves complementarity, which is beneficial but also may be impossible for some vendors due to competition.

Figure 1 presents an example, where 4 groups of friends form opposing factions, except for one node in each group which also participates in a larger central clique (weights 1 are in green, all other pairs are enemies). The number inside a node represents its current utility.

We focus on the following properties of a coloring game:

¹Although we keep weights symmetric for simplicity here, we also present results for the asymmetrical case in which an alliance is more profitable to one of the nodes (Section 4).

²While we mostly focus on the case where the utility is the sum of all these weights, we also present a more general case using subsets (see Section 4).

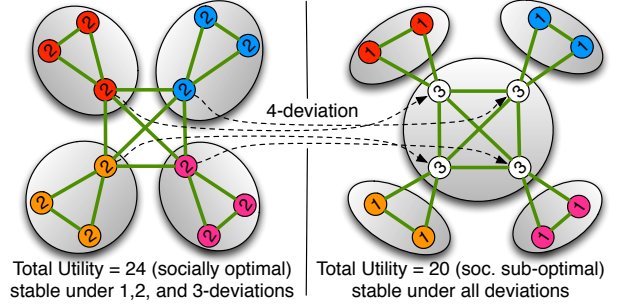


Figure 1: Example of move in a coloring game.

- **Stability:** We say that a configuration is *k-stable* if no deviation of at most size *k* exists. The existence of such configurations is necessary for the game to converge when these deviations take place. This condition ensures that groups are self-perpetuating: their members have no incentive to deviate and form new alliances.

In Figure 1, peripheral cliques are self-perpetuating when deviations of at most size 3 are allowed, but they are unstable under 4-deviations.

- **Time to converge:** When *k-stable* configurations exist, we also wish to prove that a small number of steps is needed for the game to converge. Note that nodes are not coordinated and hence we need to bound the *longest sequence* of steps before convergence.

For instance, in Figure 1, the longest sequence ending in the 4-stable configuration on the right has at least one more step than the longest sequence ending to the left.

- **Efficiency:** All games that converge end in a configuration in which no players can improve their score, but how does it compare to some optimal choice? One way to answer this question is to compare the sum of all utilities received by players to the configuration where this quantity is maximized.

In Figure 1, the socially optimal configuration is not 4-stable, but the right configuration achieves $\frac{5}{6}$ of it.

Our contribution: the uniform case.

In the case where all edges that are not conflicts have the same unit weight, a player receives a utility equal to the number of people in her group. One may think that she wishes to join a larger group, but she may be prevented from doing so by the presence of a conflict. We prove that this game is equivalent to an information sharing game introduced in [9]. The authors showed that for small values of *k* (up to *k* = 3) any arbitrary sequence of deviations converges in polynomial time. However, while the mere existence of a *k-stable* configuration has been independently established for any *k*, we still have no understanding of the *cost* in the number of deviations that are required to attain it. The potential function method used for *k* ≤ 3 fails otherwise, primarily because deviations of size 4 can decrease the overall utility of the game (*e.g.*, Figure 1). A first hint at the difficulty of this problem is that finding a *k* stable partition is NP-hard whenever *k* is part of the input.

We prove that the longest sequence of steps in a uniform game can be analyzed in the Dominance Lattice (previously introduced in theoretical physics and combinatorics). In contrast with potential function methods, it provides new and *tight* polynomial bounds when such exist. Moreover, a more detailed analysis of these sequences allows us to rigorously prove the *sudden complexity increase* occurring at $k = 4$, as the number of steps in this game admits no polynomial bound. It completes the analysis of convergence in the uniform case.

Our contribution: the general case.

Coloring games when weights are non-uniform are far more difficult to analyze, as no result exists prior to this work that guarantees even the existence of a k -stable configuration. We first generalize the potential method to prove the convergence in polynomial time for all games when only 1-deviations are allowed. We prove that this extends to 2-deviations whenever all positive weights are equal to 1. We show it also provides tight bounds on convergence time.

Remarkably, these two results complete the characterization of the general case in its entirety: Even with only 3-deviations, conflicts and binary 0-1 weights, we show games may not converge, although their utility never decreases. All other sets of weights that are neither trivial, nor equivalent to these previously mentioned, admit games that are not 2-stable.

Our results again strengthen following a *sudden complexity increase*: for any $k \geq 1$, and a fixed set of weights with conflicts, coloring games either are always k stable or it is NP hard to decide which ones are. Hence, the first occurrence of a counter-example always makes the eventual behavior of a game in general beyond prediction under polynomial.

Our contribution: more general coloring games.

We finally extend our results to account for more general classes of games. First, we show that assuming asymmetrical weights or allowing the “gossip” deviation defined in [9] immediately creates instability. Second, we show somewhat counter-intuitively that allowing nodes to choose multiple colors, while resolving all our previous counter-examples, makes stability harder to obtain *even in the uniform case*. Finally, we prove that all our results naturally extend beyond symmetric pairwise relationships to account for multimodal interaction in hypergraphs.

We conclude with two results illustrating the tension between stability and efficiency in coloring games. While the price of anarchy is generally difficult to bound, collusions and multiple colors per nodes bring natural improvement whenever the game converges. This complements our analysis by showing that the cases in which convergence is the most difficult to guarantee are also those in which the system may form better groups overall. It motivates future research to design games operating close but not beyond the complexity frontier we unveil.

Related work.

The work most relevant to coloring games is an information sharing model where groups are connected components of a graph formed as local edges are opened by peers [9]. We already discussed those results in the uniform case, for which this game is equivalent to ours. In the general case, the authors provide one counter-example of stability, which

Dynamic of the system

Input: A positive integer $k \geq 1$, a set of symmetric weights \mathcal{W} , and a graph $G = (V, E, w)$.

Output: A configuration k -stable for G .

- 1: Let C_0 be the configuration composed of $|V|$ singletons groups.
 - 2: Set $i = 0$.
 - 3: **while** there exists a k -deviation for C_i **do**
 - 4: Set $i = i + 1$.
 - 5: Compute the configuration C_i after any k -deviation.
 - 6: **return** Configuration C_i .
-

is built around the *gossip* deviation. In gossip deviation, two nodes are allowed to merge their connected components or, equivalently, that the two nodes’ colors become a single color. We analyze games with and without the gossip deviation, as we show more generally that the deviation always introduces instability.

Graph coloring, in which an edge represents such a conflict, has been the subject of a long tradition of computational research. Similarly, the structure of social networks with positive and negative edges has been a topic of interest since the 1950s. A first aspect that sets our results apart from previous works is that we consider the individual interest of the nodes, instead of attempting to satisfy a global property (*e.g.*, minimizing the number of colors, or avoiding structural imbalance created by triangles with exactly two positive edges). It is indeed easy to show that graphs that are structurally balanced are k -stable for any k , but our results apply more generally to characterize any graph. Our work also highlights the importance of accounting for complexity, and not only existential results, when analyzing networks with positive and negative edges.

This model of group formation is a novel direction. Most models explaining network formation either deal with ensuring connectivity [6], or matching [11] under local cost constraint such as number of edges or queries. *Alternatively*, they may balance a global risk propagation with rewards from local connections [1]. In contrast, we *jointly* consider both the positive and negative aspects of social interaction *within the same graph*. This mirrors the situation many users face online when balancing privacy with utility. This topic, one of fundamental importance to the design of social networking software (see *e.g.*, [7]) was previously unexplored. Our work is also one of the first work to consider how groups and connections form when users follow non-uniform preferences, and to attempt to go beyond Nash equilibrium, as opposed to most previous game-theoretical approaches [4].

2. UNIFORM CASE: LONGEST SEQUENCE

In *uniform* coloring games, we assume that, except for conflict edges, all edges have the same unit weight. Alternatively, this game is entirely characterized by an unweighted and undirected *conflict graph* that contains all the conflict edges. As shown in [9] k -stable partitions always exist for any value of k , but the analysis of the game is left open as soon as 4-deviations are allowed.

We first provide an argument why this game converges for any value of k . Let us define the *configuration vector*

of a partition C as $v(C) = (v_1, \dots, v_n)$, where for any $i = 1, \dots, n$, v_i denotes the number of groups of size i . One can see that a deviation of any size produces a configuration vector that is strictly higher in the lexicographical ordering. The game always converges in a finite number of steps.

LEMMA 1. (Convergence) For any $k \geq 1$, for any graph $G = (V, E)$, the system converges to a k -stable partition.

PROOF. Let C_i, C_{i+1} be two partitions for G such that C_{i+1} is obtained from C_i after a k -deviation for C_i . We prove $v(C_i) <_L v(C_{i+1})$ where L is the lexicographical order. By definition, for any $u \in S$, we have $f_u(C_i) < f_u(C_{i+1})$ where S represents the set of nodes involving in the k -deviation ($|S| \leq k$). Thus, we get $v(C_{i+1}) - v(C_i) = (0, \dots, 0, k, \dots)$, and so $v(C_i) <_L v(C_{i+1})$. Finally, as the number of possible vectors is finite, we obtain the convergence of the system. \square

We can then define $L(k, n)$ as the size of a longest sequence of k -deviations among all the graphs with at most n nodes. One can prove that $L(k, n)$ is always attained in the graph containing no conflict edges, which will be instrumental in future proofs. We define $L(k, G)$ as the longest sequence of k -deviations for a given graph G .

LEMMA 2. For any $k, n \geq 1$, $L(k, n) = L(k, G^0)$ where $G^0 = (V, E)$ is such that $|V| = n$ and $|E| = 0$.

PROOF. Let $k, n \geq 1$. Let $G = (V, E)$ be any graph with $|V| = n$. We prove that $L(k, G^0) \geq L(k, G)$. Let S be a longest sequence for G of size $L(k, G)$. We can mimic the sequence S for G in order to get a valid sub-sequence S^0 for G^0 . Indeed, every k -deviation of S is a valid deviation for G^0 . It might be possible that, at the end of sequence S^0 , the current partition is not k -stable for G^0 . Thus, some extra k -deviations might be necessary. We get $L(k, G^0) \geq L(k, G)$ and so $L(k, n) = L(k, G^0)$. \square

Table 1 summarizes our contributions: the exact value of $L(k, n)$ for small values of k and the proof that its combinatorial properties suddenly increases as $k = 4$.

k	Best known	Our results	
1	$O(n^2)$ [9]	$\sim \frac{2}{3}n^{3/2}$	Theorem 1
2	$O(n^2)$ [9]	$\sim \frac{2}{3}n^{3/2}$	Theorem 2
3	$O(n^3)$ [9]	$\Omega(n^2)$	Lemma 4
≥ 4	$O(2^n)$ [9]	$\Omega(n^{a \ln(n)}), O(e^{\sqrt{n}})$	Theorem 3

Table 1: Our contributions for $L(k, n)$ values.

2.1 Exact value for $k < 3$

In [9], the authors proved the global utility increases for each k -deviation when $k \leq 2$. As that potential function is also upper bounded by $O(n^2)$, the system converges to a 2-stable partition in at most a quadratic time.

We improve this result as we completely solve this case and give the exact (non-asymptotic) value of $L(k, n)$ when $k \leq 2$. It also proves that this potential function method is not tight. The gist of the proof is to re-interpret sequences of deviations in the Dominance Lattice. This object has been widely used in theoretical physics and combinatorics to study systems in which the addition of one element (e.g., a grain of sand) creates consequences in cascade (e.g., the re-configuration of a sand pile) [5]. Let us first define:

DEFINITION 1 ([2]). A partition of $n \geq 1$, is a sequence $l_1 \geq l_2 \geq \dots \geq l_n \geq 0$ of integers such that $\sum_{i=1}^n l_i = n$.

Given any graph with n nodes, there are as many vectors of configurations as there are partitions of the integer n . If we denote it by p_n , by lexicographical ordering we know the system reaches a stable configuration in at most $p_n = O(e^{\sqrt{n}})$ steps. This is already far less than 2^n , which was shown to be the best upper bound that can be obtained for $k \geq 4$ when using an additive potential function [9].

To go further, one needs to go beyond lexicographical ordering. For $n \geq 6$ one can see that some configurations may never be in the same sequence. It is hence important to deal with a partial ordering instead of a total one.

DEFINITION 2. (dominance ordering) Given two partitions of $n \geq 1$, $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_n$, we say that a dominates b if $\sum_{j=1}^i a_j \geq \sum_{j=1}^i b_j$ for all $1 \leq i \leq n$.

Brylawski proved that this ordering creates a lattice ([2]) called the Dominance Lattice. Successors and predecessors can be defined by covering relation:

DEFINITION 3. (covering) Given two partitions of $n \geq 1$, $a_1 \geq \dots \geq a_n$ and $b_1 \geq \dots \geq b_n$, a covers b if and only if a dominates b and there is no other integer partition c such that a dominates c and c dominates b .

There is a strong relation between this structure and 1-deviation, that we can illustrate as follows:

LEMMA 3 ([2]). Partition a covers b if, and only if, there exist j, k such that: (i) $a_j = b_j + 1$; (ii) $a_k = b_k - 1$; (iii) for all l such that $l \neq j, l \neq k$, we have $a_l = b_l$; (iv) either $k = j + 1$ or $b_j = b_k$.

COROLLARY 1. Given a graph $G^0 = (V, E)$, $|V| = n$ and $|E| = 0$, let a, b be two integer partitions of n . Then a dominates b if, and only if, there exist two configurations C_1, C_2 for G such that $v(C_1) = b$, $v(C_2) = a$, and there is a sequence of 1-deviations from C_1 to C_2 .

PROOF. First, assume a covers b . By Lemma 3, there exist j, k such that $a_j = b_j + 1$, $a_k = b_k - 1$, and for all i such that $i \neq j, k$, $a_i = b_i$. Moreover, $b_j \geq b_k$, since $k = j + 1$ or $b_j = b_k$. Let C_1 be any configuration of G such that $v(C_1) = b$. We can suppose there are n groups in C_1 by adding empty groups. We order those groups by decreasing order on their size, denoted $C_1 = \{V_1, \dots, V_n\}$. Now let u be any vertex in V_k . Such a vertex exists because $|V_k| = b_k > 0$. Then u can move from V_k to V_j , and it is a valid 1-deviation because $|V_j| = b_j \geq b_k$. In so doing, we get a configuration $C_2 = \{V'_1, \dots, V'_n\}$ such that $|V'_j| = |V_j| + 1$, $|V'_k| = |V_k| - 1$, and for all i such that $i \neq j, k$, $|V'_i| = |V_i|$. In other words, $v(C_2) = a$.

More generally, if a dominates b then there exists a sequence $a = q_1, q_2, \dots, q_p = b$, such that for every $1 \leq i \leq p - 1$, q_i covers q_{i+1} . Therefore, we can iterate the process, and we get the expected result.

Conversely, let C_1, C_2 be two configurations of G , such that there is a 1-deviation from C_1 to C_2 . Again, we order the groups by decreasing size, denoted respectively $C_1 = \{V_1, \dots, V_n\}$ and $C_2 = \{V'_1, \dots, V'_n\}$. Let $S = \{u\}$ be the 1-deviation that breaks C_1 . Then u leaves some group V_k for another group V_j . Furthermore, $|V_j| \geq |V_k|$ by the hypothesis. So, we get either $|V_j| = |V_k|$, or $|V_j| > |V_k|$, hence $j \leq k - 1$. Let us denote $a = v(C_2)$ and $b = v(C_1)$. By a

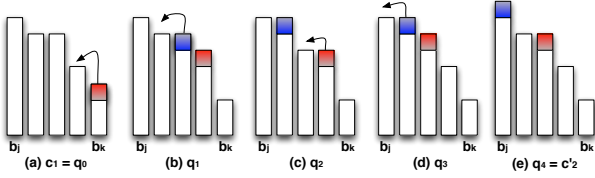


Figure 2: An example of decomposition for a 1-deviation from group b_k to group b_j .

reordering of groups with equal size, we may assume that V_j is the first group in decreasing order with size $|V_j|$, whereas V_k is the last group in decreasing order with size $|V_k|$.

- if $g \in \{1, \dots, j-1\}$, then $\sum_{i=1}^g a_i = \sum_{i=1}^g b_i$;
- if $g \in \{j, \dots, k-1\}$, then $\sum_{i=1}^g a_i = [\sum_{i=1}^g b_i] + 1$;
- if $g \in \{k, \dots, n\}$, then $\sum_{i=1}^g a_i = \sum_{i=1}^g b_i$.

As a consequence, we have that a dominates b by the hypothesis. \square

In other words, any sequence of 1-deviations from a partition C_1 to another partition C_2 can be decomposed into more elementary 1-deviations. Note that, in so doing, we may get another final configuration $C'_2 \neq C_2$ but it will have the same configuration vector, as seen on an example in Figure 2. For our purpose, since the new sequence has a size that is larger than the original one, and it is a sequence of 1-deviations that is also valid, it allows us to conclude.

THEOREM 1. $L(1, n) = 2\binom{m+1}{3} + mr$, where r, m are unique solutions of $n = \frac{m(m+1)}{2} + r$, $0 \leq r \leq m$. This implies that $L(1, n) \sim \frac{2}{3}n\sqrt{n}$ as n gets large.

PROOF. Let C_1, C_2, \dots, C_p be a sequence of configurations obtained by 1-deviations. By Corollary 1, the sequence $v(C_1), v(C_2), \dots, v(C_p)$ is a chain, and so $L(1, n)$ is upper-bounded by the size of a longest chain in the Dominance lattice. Conversely, let q_1, q_2, \dots, q_p be any chain of integer partitions of n . By the proof of Corollary 1, we have that there exists a sequence of partitions C_1, \dots, C_p such that for every $1 \leq i \leq p$ $v(C_i) = q_i$, and for every $1 \leq i \leq p-1$ there is a sequence of 1-deviations from C_i to C_{i+1} . Hence $L(1, n)$ is lower-bounded by the size of a longest chain of integer partitions. \square

Interestingly, the analysis is exactly the same when 2-deviations are added. Following a similar line of argument using vectors of configuration we can further prove:

THEOREM 2. $L(2, n) = L(1, n)$.

PROOF. Clearly, $L(2, n) = \Omega(L(1, n))$.

Let $G^0 = (V, E)$ be a graph such that $|V| = n$ and $|E| = 0$. Let C be any configuration for G^0 . Suppose that $S = \{u, v\}$ breaks C . We denote by V_u, V_v the original group of u, v , respectively, and by $V_{u,v}$ the group both vertices agree to join together. If $|V_{u,v}| \geq |V_u|$ or $|V_{u,v}| \geq |V_v|$, then the 2-deviation can be decomposed into 1-deviations. So we assume that $|V_{u,v}| = |V_u| - 1 = |V_v| - 1$. There are two cases.

a) Assume $V_u = V_v$. After the 2-deviation, we replace $V_u, V_{u,v}$, by $V_u \setminus \{u, v\}, V_{u,v} \cup \{u, v\}$, that is we get two

groups of size $|V_u| - 2, |V_u| + 1$, respectively, instead of two groups of size $|V_u| - 1, |V_u|$, respectively. Thus, if a vertex of $V_{u,v}$ breaks the configuration C and joins group V_u , then we get the same vector of configuration.

b) Now assume $V_u \neq V_v$. After the 2-deviation, we replace $V_u, V_v, V_{u,v}$, by $V_u \setminus \{u\}, V_v \setminus \{v\}, V_{u,v} \cup \{u, v\}$, that is we get three groups of size $|V_u| - 1, |V_u| - 1, |V_u| + 1$, respectively, instead of three groups of size $|V_u| - 1, |V_u|, |V_u|$, respectively. Thus, if a vertex of V_v breaks the configuration C and joins group V_u , then we also get the same vector of configuration.

Finally, any vector of configuration that is obtained from a 2-deviation may also be gotten from a sequence of 1-deviation. Thus $L(2, n) = L(1, n) = \theta(n\sqrt{n})$. \square

2.2 Non polynomial lower bound for $k > 3$

As seen above, we already showed that $L(4, n) = O(e^{\sqrt{n}})$, which is already much better than what additive potential function methods can attain [9]. However, the computational cost of 4-deviations is unknown. We resolve this problem by proving $L(4, n)$ cannot be bounded polynomially.

THEOREM 3. $L(4, n) = \Omega(n^{a \ln(n)})$ where $a > 0$.

PROOF. Let $G^0 = (V, E)$ with $|E| = 0$. Suppose $n = |V| = cL$ where c, L are positive integers. We first construct a partition C^0 for G^0 for which the vector of partition $v(C^0) = (v_1, v_2, \dots, v_n)$ is such that for any j , $1 \leq j \leq L$, $v_j = c$, and for any j , $L+1 \leq j \leq n$, $v_j = 0$. Clearly this partition can be obtained with a sequence of 1-deviations. In the following, we only consider groups of sizes at most L .

We introduce the notion of *reverse vector of difference*. Starting from partition C^0 , we define the initial reverse vector of difference as $v^0 = (v^0(1), v^0(2), \dots, v^0(L))$ where, for any j , $1 \leq j \leq L$, $v^0(j) = v_{L-j+1} - c$ represents the number of groups of size $L-j+1$ minus the initial number c . The vector of difference v for a configuration C , is such that, for any j , $1 \leq j \leq L$, $v(j)$ represents the difference between the number of groups of size $L-j+1$ that have been created and the number of groups of size j that have been removed, in the sequence of deviations from v^0 to v .

A reverse vector of difference v is *c-balanced* if, for any j , $1 \leq j \leq L$, $v(j) \geq -c$. Any sequence of deviations must induce a c -balanced vector (there are exactly c groups of each size at most L at the beginning).

We only consider 4-deviations that consist in creating five groups of sizes $p, p-2, p-2, p-2, p-2$, and removing five groups of sizes $p-4, p-1, p-1, p-1, p-1$ for some p , $5 \leq p \leq L$. We consider such a 4-deviation because it strictly decreases the global utility and so it will allow us to construct long sequences. We define an elementary vector of difference ϕ_{L-p+1} of size L : $\phi_{L-p+1} = (\dots, 1, -4, 4, 0, -1, \dots)$ where dots represent 0 values, 1 corresponds to the group of size p that has been created, -4 corresponds to the four groups of size $p-1$ that have been removed, etc. The first 1 at index $L-p+1$ means that a group of $L-p+1$ has been created, -4 is at index $L-p+2$, etc. Given a reverse vector of difference v (corresponding to a current partition C), the reverse vector of difference v' for C' obtained after a 4-deviation for C , is $v' = v + \phi_{L-p+1}$.

In other words,

$$\phi_{L-p+1}(j) = \begin{cases} 1 & \text{if } j = L - p + 1 \\ -4 & \text{if } j = L - p + 2 \\ 4 & \text{if } j = L - p + 3 \\ -1 & \text{if } j = L - p + 5 \\ 0 & \text{otherwise} \end{cases}$$

We define notations for 1-deviations that consist in creating two groups of sizes p, q , and removing two groups of sizes $p-1, q+1$, for some $p, q, 3 \leq q+2 \leq p \leq L$. We define an elementary vector of difference $\phi_{L-p+1, L-q+1}$ of size L :

$$\phi_{L-p+1, L-q+1} = \begin{cases} (\dots, 1, -1, \dots, -1, 1, \dots) & \text{if } p > q + 2 \\ (\dots, 1, -2, 1, \dots) & \text{if } p = q + 2 \end{cases}$$

where the first 1 (index $L - p + 1$) corresponds to the group of size p that has been created. The second value 1 (index $L - q + 1$) corresponds to the group that has been created.

In other words,

$$\phi_{L-p+1, L-q+1}(j) = \begin{cases} 1 & \text{if } j = L - p + 1 \\ -1 & \text{if } j = L - p + 2 \\ -1 & \text{if } j = L - q \\ 1 & \text{if } j = L - q + 1 \\ 0 & \text{otherwise} \end{cases}$$

We now define a particular sequence of 1-deviations. For any j_1, j_2, d such that $1 \leq j_1 < j_1 + d \leq j_2 - d < j_2 \leq L$, we define $\phi_{j_1, j_1+d, j_2-d, j_2} = \sum_{j=0}^{d-1} \phi_{j_1+j, j_2-j} =$

$$\begin{cases} (\dots, 1, \dots, -1, \dots, -1, \dots, 1, \dots) & \text{if } j_1 + d \neq j_2 - d \\ (\dots, 1, \dots, -2, \dots, 1, \dots) & \text{if } j_1 + d = j_2 - d \end{cases}$$

Given a reverse vector of difference v and an integer $r, 1 \leq r \leq L$, we define $v' = \Phi(v, r)$ as follows:

$$\begin{cases} v'(j + r - 1) = v(j) & \forall j, 1 \leq j \leq L - r + 1, \\ v'(j) = 0 & \forall j, 1 \leq j < r. \end{cases}$$

The size s of a reverse vector of difference v is such that $v(s) \neq 0$ and $v(j) = 0$ for any $j, s+1 \leq j \leq L$.

We define the notion of *symmetric vector*. A reverse vector of difference v is symmetric if for any $j, 1 \leq j \leq s, v(j) = v(s - j + 1)$ where s is the size of v .

CLAIM 1. *Let v be any symmetric vector of difference of size s . Then, for any positive integers r and $d, 1 + (r-1)d \leq s, v' = \sum_{h=0}^{r-1} \Phi(v, 1 + hd)$ is a symmetric vector.*

PROOF. Vector v' has size $s' = (r-1)d + s$. Let first suppose that r is even. $v' = \sum_{h=0}^{r/2-1} \Phi(v, 1 + hd) + \sum_{h=r/2}^{r-1} \Phi(v, 1 + hd)$. We prove that, for any $r', 0 \leq r' \leq r/2 - 1, y_{r'} = \sum_{h=0}^{r'} \Phi(v, 1 + hd) + \sum_{h=r-1-r'}^{r-1} \Phi(v, 1 + hd)$ is a symmetric vector. Note that the size of $y_{r'}$ is $s' = (r-1)d + s$ for any $r', 0 \leq r' \leq r/2 - 1$.

By induction on r' . Suppose $r' = 0$. We have $y_0 = \Phi(v, 1) + \Phi(v, 1 + (r-1)d)$. For any $j, 1 \leq j \leq (r-1)d + s, y_0(j) = v(j) + v(j - (r-1)d)$. As v is a symmetric vector of size s , then $v(j - (r-1)d) = v(s - j + (r-1)d + 1)$, and so $v(j - (r-1)d) = v(s' - j + 1)$ because y_0 has size s' . Thus, we get $y_0(j) = v(j) + v(s' - j + 1)$. We deduce that $y_0(s' - j + 1) = v(s' - j + 1) + v(s' - (s' - j + 1) + 1)$, and so $y_0(s' - j + 1) = v(s' - j + 1) + v(j)$. As a result, y_0 is a symmetric vector.

Suppose it is true for $r', 0 \leq r' \leq r-2$, we prove it is also true for $r' + 1$. Vector $y_{r'}$ is a symmetric vector. We have $y_{r'+1} = y_{r'} + \Phi(v, 1 + (r' + 1)d) + \Phi(v, 1 + (r - r' - 2)d)$. First, if $j \leq (r+1)d$, then $y_{r'+1}(j) - y_{r'}(j) = 0$ by construction. Also, $y_{r'+1}(j) - y_{r'}(j) = 0$ for any $j, (r - r' - 2)d + s \leq j \leq s'$. For any $j, 1 \leq j \leq (r-1)d + s, y_{r'+1}(j) - y_{r'}(j) = v(j - (r' + 1)d) + v(j - (r - r' - 2)d)$. Let $x_1 = (r+1)d + 1$ and let $x_2 = (r - r' - 2)d + s - 1$. Consider the sub-vector $y'_{r'}$ of $y_{r'}$ defined as follows: $y'_{r'} = (y_{r'}(x_1), y_{r'}(x_1 + 1), \dots, y_{r'}(x_2 - 1), y_{r'}(x_2))$. Clearly $y'_{r'}$ is a symmetric vector because $x_1 - 1 = s' - x_2$. Thus, we can apply proof for case $r' = 0$ to show that $y'_{r'+1} = \Phi(v, 1 + (r' + 1)d) + \Phi(v, 1 + (r - r' - 2)d)$ is a symmetric vector. So $y_{r'+1}$ is a symmetric vector.

Finally, for any $r', 0 \leq r' \leq r/2 - 1, y_{r'}$ is a symmetric vector. We get that v' is a symmetric vector.

Consider now that r is odd. We set $v'' = \sum_{h=0}^{(r-1)/2-1} \Phi(v, 1 + hd) + \sum_{h=(r-1)/2+1}^{r-1} \Phi(v, 1 + hd)$. Using the previous induction, we prove that v'' is symmetric vector. We now prove that $v' = v'' + \Phi(v, 1 + d(r-1)/2)$ is a symmetric vector. Let $v''' = \Phi(v, 1 + d(r-1)/2)$. We have $v'''(j) = v'''((r-1)d + s - j)$ because $d \frac{r-1}{2} + 1 - 1 = d(r-1) + s - d \frac{r-1}{2} - s$. Finally, v' is a symmetric vector because v'' is symmetric. \square

Let $l(1), T \geq 1$ be any two integers such that $2^{T-1}l(1) \leq L$. We will explain this choice at the end of the proof. For any $i, 1 \leq i \leq T-1$, we construct the vector v^{i+1} from v^i such that these vectors have a special property, called *Good Property* (Definition 4).

We first construct v_1 . Let $t = l(1) - 1$.

We now construct $z^1 = (\sum_{j=1}^t \phi_j) + \phi_{2,3,3,4} + \phi_{t+1,t+2,t+2,t+3} + \phi_{t+4,t+5,t+6,t+7} + \phi_{2,4,4,6}$. For any $j, 1 \leq j \leq L$:

$$z^1(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 2 \\ -1 & \text{if } j = 3 \\ 1 & \text{if } j = 6 \\ 1 & \text{if } j = t + 2 \\ -1 & \text{if } j = t + 5 \\ -1 & \text{if } j = t + 6 \\ 1 & \text{if } j = t + 7 \\ 0 & \text{if } j \notin \{1, 2, 3, 6, t+2, t+5, t+6, t+7\} \end{cases}$$

We now construct $z^2 = \sum_{j=1}^{t-4} \Phi(z^1, j)$. For any $j, 1 \leq j \leq$

L :

$$z^2(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 3 \\ -1 & \text{if } j = 4 \\ -1 & \text{if } j = 5 \\ -1 & \text{if } j = t - 3 \\ 1 & \text{if } j = t - 1 \\ 1 & \text{if } j = t \\ 1 & \text{if } j = t + 1 \\ 1 & \text{if } j = t + 2 \\ 1 & \text{if } j = t + 3 \\ 1 & \text{if } j = t + 4 \\ -1 & \text{if } j = t + 6 \\ -1 & \text{if } j = 2t - 2 \\ -1 & \text{if } j = 2t - 1 \\ -1 & \text{if } j = 2t \\ 1 & \text{if } j = 2t + 2 \\ 0 & \text{otherwise} \end{cases}$$

We finally construct $v^1 = z^2 + \phi_{5,t,t+3,2t-2} + \phi_{t-3,t-1,t+4,t+6}$.
For any j , $1 \leq j \leq L$:

$$v^1(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 3 \\ -1 & \text{if } j = 4 \\ 1 & \text{if } j = t + 1 \\ 1 & \text{if } j = t + 2 \\ -1 & \text{if } j = 2t - 1 \\ -1 & \text{if } j = 2t \\ 1 & \text{if } j = 2t + 2 \\ 0 & \text{otherwise} \end{cases}$$

The intuitive idea is that such consecutive 4-deviations, called *cascade*, balance the vector of difference. v^1 has the Good Property and Claim 2 shows that each vector of difference from v^0 to v^1 is c_1 -balanced (for some constant c_1).

For any i , $1 \leq i \leq T - 1$, we denote by s^i the size of the reverse vector of difference v^i , and set $l(i) = s^i/2$. Note that the size of v^1 depends on the choice of $l(1)$, $s^1 = 2l(1)$.

DEFINITION 4. v^i has the Good Property if s^i is even and if there exists t_1^i, t_2^i with $1 < t_1^i < t_2^i < 2t_1^i - 1$, $t_2^i \leq 2^i$, such that for any j , $1 \leq j \leq L$:

$$v^1(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = t_1^i \\ -1 & \text{if } j = t_2^i \\ 1 & \text{if } j = l(i) \\ 1 & \text{if } j = l(i) + 1 \\ -1 & \text{if } j = 2l(i) - t_2^i + 1 \\ -1 & \text{if } j = 2l(i) - t_1^i + 1 \\ 1 & \text{if } j = 2l(i) \\ 0 & \text{otherwise} \end{cases}$$

CLAIM 2. There exists a constant c_1 such that every reverse vector of difference from v^0 to v^1 , is c_1 -balanced.

In order to build v^{i+1} from v^i , we define an intermediary construction. For any i , $1 \leq i \leq T - 1$, we define $u^{i+1} =$

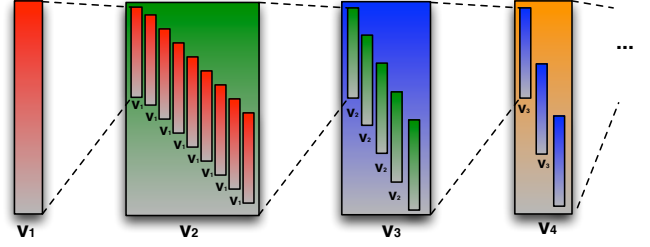


Figure 3: Long sequence using recursive cascades.

$\sum_{j=0}^{\alpha(i)} \Phi(v^i, 1+j(t_1^i-1))$ where $\alpha(i)$ is the largest even integer j such that $t_2^i + j(t_1^i - 1) < l(i)$. We now prove that if v^i has the Good Property, then it is possible to construct u^{i+1} and v^{i+1} such that v^{i+1} has the Good Property. We will prove in Claim 4 that our constructive proof makes each vector of difference c -balanced (for some c defined later).

Figure 3 represents the construction of u^2 (and v^2) from the vector v^1 . The vector v^1 is applied a larger number of times and the starting point is shifted in order to balance the vectors of differences. This cascade technique is used recursively. The vector v^3 is obtained by applying a cascade of vectors v^2 , and so on. We prove that this construction uses a large number (non polynomial) of deviations and as the vector will be enough balanced, the number of nodes will remain small compared to the number of deviations.

CLAIM 3. For any i , $1 \leq i \leq T - 1$, if v^i has the Good Property, then there exists a sequence of deviations from u^{i+1} to v^{i+1} such that v^{i+1} has the Good Property.

PROOF. For some i , $1 \leq i \leq T - 1$, suppose that v^i has the Good Property. By construction there exists t_1^{i+1} and t_2^{i+1} such that, for any j , $1 \leq j \leq t_2^{i+1}$:

$$u^{i+1}(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = t_1^{i+1} \\ -1 & \text{if } j = t_2^{i+1} \\ 0 & \text{otherwise} \end{cases}$$

where $t_1^{i+1} = t_2^i$ and $t_2^{i+1} = 2t_1^i - 1$.

We get $1 < t_1^{i+1} < t_2^{i+1}$ because $1 < t_2^i < 2t_1^i - 1$ by hypothesis. Also, $t_2^{i+1} < 2t_1^{i+1} - 1$ because $2t_1^i - 1 < 2t_2^i - 1$. Finally, $t_2^{i+1} \leq 2^{i+1}$ because $t_2^{i+1} < 2t_1^{i+1} = 2t_2^i \leq 2^i \cdot 2$. By Claim 1, we obtain the properties for the end of the vector.

Note that $l(i+1) \leq 2l(i)$. Furthermore $s^{i+1} = \alpha(i)(t_1^i - 1) + s^i$. As $\alpha(i)$ and s^i are even integers, then s^{i+1} is even.

By Claim 1, we get that u^{i+1} is a symmetric vector, that is for any j , $1 \leq j \leq 2l(i+1)$, $u^{i+1}(j) = u^{i+1}(2l(i+1) - j + 1)$. Furthermore, by the choice of $\alpha(i)$, we get that there exist j_1, j_2 , $t_2^{i+1} < j_1 < j_2 < 2l(i) - t_1^i + 1$, such that for any j , $t_2^{i+1} \leq j \leq 2l(i) - t_1^i + 1$:

$$u^{i+1}(j) \in \begin{cases} \{-1, 0\} & \text{if } t_2^{i+1} < j \leq j_1 \\ \{0, 1\} & \text{if } j_1 < j < j_2 \\ \{-1, 0\} & \text{if } j_2 \leq j < 2l(i) - t_1^i + 1 \end{cases}$$

Furthermore:

$$\sum_{j=t_2^{i+1}+1}^{j_1} u^{i+1}(j) + \sum_{j=j_2}^{2l(i)-t_1^i} u^{i+1}(j) = -\sum_{j=j_1+1}^{j_2-1} u^{i+1}(j).$$

As $u^i(l(1)) = 1$, $u^{i+1}(l(i) + \frac{\alpha(i)}{2}(t_1^i - 1)) = 1$. Also, $u^{i+1}(l(i) + 1 + \frac{\alpha(i)}{2}(t_1^i - 1)) = 1$. As $\alpha(i)$ is even, we get

$u^{i+1}(l(i+1)) = 1$ because $l(i) + \frac{\alpha(i)}{2} = \frac{s^{i+1}}{2}$. Recall that $s^{i+1} = \alpha(i)(t_1^i - 1) + s^i$. Thus, $u^{i+1}(l(i+1) + 1) = 1$.

We now prove that there exists a sequence of 1-deviations in order to construct v^{i+1} with the Good Property. As u^{i+1} is a symmetric vector (Claim 1), if $u^{i+1}(j) = -1$ for any j , $t_2^{i+1} < j \leq j_1$, then $u^{i+1}(2l(i+1) - j + 1) = -1$. Let j such that $u^{i+1}(j) = -1$ and $t_2^{i+1} < j \leq j_1$. By previous remark and again by symmetry, there exists j' , $j_1 < j' < j_2$, $j' \notin \{l(i+1), l(i+1) + 1\}$, such that $u^{i+1}(j') = u^{i+1}(2l(i+1) - j' + 1) = 1$.

We construct $z_1^{i+1} = u^{i+1} + \phi_{j,j',2l(i+1)-j'+1,2l(i+1)-j+1}$. This construction clearly keeps the symmetry property. As $\sum_{j=t_2^{i+1}+1}^{j_1} u^{i+1}(j) + \sum_{j=j_2}^{2l(i)-t_1^i} u^{i+1}(j) = -\sum_{j=j_1+1}^{j_2-1} u^{i+1}(j)$, then we can find another 4-uplet $(j, j', 2l(i+1) - j' + 1, 2l(i+1) - j + 1)$, $j' \notin \{l(i+1), l(i+1) + 1\}$, to construct $z_2^{i+1} = z_1^{i+1} + \phi_{j,j',2l(i+1)-j'+1,2l(i+1)-j+1}$. And so on until $v^{i+1} = z_f^{i+1}$ where $f = \frac{1}{2} \sum_{j=j_1+1}^{j_2-1} u^{i+1}(j)$.

The vector of difference v^{i+1} has the Good Property. \square

As v^1 has the Good Property, we deduce that for any i , $1 \leq i \leq T$, v^i has the Good Property.

CLAIM 4. For any i , $1 \leq i \leq T$, any vector v of difference in the sequence from v^0 to v^i , is $(c_1 + i - 1)$ -balanced where c_1 is the constant defined in Claim 2.

PROOF. We prove the result by induction on i . The vector v^0 is 0-balanced by definition. Claim 2 proves the result for $i = 1$. Suppose it is true for any $i' \leq i \leq T - 1$. We prove that it is true for $i + 1$. We construct v^{i+1} from v^i as described in the proof of Claim 3. Recall that we first build $u^{i+1} = \sum_{j=0}^{\alpha(i)} \Phi(v^i, 1 + j(t_1^i - 1))$ where $\alpha(i)$ is the largest even integer j such that $t_2^i + j(t_1^i - 1) < l(i)$.

For any b , $0 \leq b \leq \alpha(i) - 1$, consider $y_b^{i+1} = \sum_{j=0}^b \Phi(v^i, 1 + j(t_1^i - 1))$. We prove that each vector of difference from v^i to y_b^{i+1} is $(c_1 + i)$ -balanced. By induction on b . It is true for $b = 0$. Indeed by the first induction hypothesis, each vector of difference from v^0 to $\Phi(v^i, 1)$, is $(c_1 + i - 1)$ -balanced. Suppose it is true for any $b' \leq b \leq \alpha(i) - 1$. We prove it is true for $b + 1$. By Claim 3, y_b^{i+1} is 1-balanced because $\Phi(v^i, 1)$ is 1-balanced (the values of the vector belong to the set $\{-1, 0, 1\}$). By definition $y_{b+1}^{i+1} = y_b^{i+1} + \Phi(v^i, 1 + (b + 1)(t_1^i - 1))$. By the first induction hypothesis, each sequence of $\Phi(v^i, 1 + (b + 1)(t_1^i - 1))$ is $(c_1 + i - 1)$ -balanced. Since y_b^{i+1} is 1-balanced, each vector of difference from y_b^{i+1} to y_{b+1}^{i+1} , is $(c_1 + i)$ -balanced. Thus, u^{i+1} is $(c_1 + i)$ -balanced.

To conclude, it remains to prove that each vector of difference from u^{i+1} to v^{i+1} , is $(c + i)$ -balanced. Consider the last 1-deviations described in the proof of Claim 3. Every deviation consists in replacing values -1 and 1 by 0 . Thus, v^{i+1} is $(c + i)$ -balanced. \square

For any i , $1 \leq i \leq T$, let S_i be the sequence of deviations from v^0 to v^i .

CLAIM 5. For any i , $1 \leq i \leq T - 1$, $l(i) \leq 2^{i-1}l(1)$.

PROOF. By induction on i . It is clearly true for $i = 1$. Suppose it is true for any $i' \leq i$. We prove it is true for $i + 1$. By construction, $l(i + 1) \leq 2l(i)$. Thus, $l(i + 1) \leq 2^{i-1}2l(1)$ by induction hypothesis. Finally we get $l(i + 1) \leq 2^i l(1)$. \square

CLAIM 6. For any i , $1 \leq i \leq T - 1$, $|S_{i+1}| \geq (\frac{l(i)}{2^i} - 3)|S_i|$.

PROOF. $|S_{i+1}| = \alpha(i)|S_i|$ with $\alpha(i)$ is the largest even integer j such that $t_2^i + j(t_1^i - 1) < l(i)$. Thus $\alpha(i) \geq \lfloor \frac{l(i) - 2^{i+1} + 1}{2^i - 1} \rfloor$ because $t_1^i, t_2^i \leq 2^i$. So $|S_{i+1}| \geq (\frac{l(i)}{2^i} - 3)|S_i|$. \square

CLAIM 7. For any i , $1 \leq i \leq T - 1$, and for any $l(1) > 2^{i+1}3$, then $|S_i| \geq \frac{l(1)^{i-1}|S_1|}{2^i}$.

PROOF. First note that $l(1) \geq l(i)$ for any $i \geq 1$. As $l(1) > 2^{i+1}3$, we get that $\frac{l(i)}{2^i} - 3 \geq \frac{l(i)}{2^{i+1}}$. By Claim 6, we get $|S_{i+1}| \geq \frac{l(i)}{2^{i+1}}|S_i|$. Thus, $|S_i| \geq \frac{l(1)^{i-1}|S_1|}{2^i}$. \square

We proved that $|S_T| \geq \frac{l(1)^{T-1}|S_1|}{2^T}$ and $l(T) \leq 2^{T-1}l(1)$.

By Claim 4, $c = c_1 + T - 1$ guarantees the feasibility of the sequence of deviations from v^0 to v^T . Thus, $n = 2^{T-1}l(1)(c_1 + T - 1)$ is sufficient to construct a valid sequence. For any $l(1)$, set $T = \lfloor \log_2(l(1)/6) \rfloor - 1$. We choose $L \geq 2^{T-1}l(1)$ as consequence. We obtain that for any n , there exists a sequence of 1 and 4-deviations S of length $\Omega(n^{a \log_2(n)})$ for some constant $a > 0$. \square

We prove a lower bound when $k = 3$ using the same techniques. Obviously, we cannot apply cascades a large number of times since any sequence of 3-deviations is of size $O(n^3)$.

LEMMA 4. $L(3, n) = \Omega(n^2)$.

PROOF. This proof is similar to the proof for $k = 4$ but it is simpler since we construct only v^1, v^2, v^3, v^4 .

We only consider here 3-deviations that consist in creating 1 group of size p , removing 3 groups of size $p - 1$, creating 3 groups of $p - 2$, and removing 1 group of size $p - 3$, for some p . Let ϕ_{L-p+1} be the vector of difference corresponding to a 3-deviation defined as follows. For any j , $1 \leq j \leq L$:

$$\phi_j = \begin{cases} 1 & \text{if } j = L - p + 1 \\ -3 & \text{if } j = L - p + 2 \\ 3 & \text{if } j = L - p + 3 \\ -1 & \text{if } j = L - p + 4 \\ 0 & \text{otherwise} \end{cases}$$

We start from the vector of difference $v^0 = (0, 0, \dots)$.

Set $v^1 = \sum_{i=1}^t \phi_i$. For any j , $1 \leq j \leq L$,

$$v^1(j) = \begin{cases} 1 & \text{if } j = 1 \\ -2 & \text{if } j = 2 \\ 1 & \text{if } j = 3 \\ -1 & \text{if } j = t + 1 \\ 2 & \text{if } j = t + 2 \\ -1 & \text{if } j = t + 3 \\ 0 & \text{otherwise} \end{cases}$$

Set $v^2 = \sum_{i=1}^{t-2} \Phi(v^1, i)$. For any j , $1 \leq j \leq L$,

$$v^2(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 2 \\ -1 & \text{if } j = t - 1 \\ 1 & \text{if } j = t \\ -1 & \text{if } j = t + 1 \\ 1 & \text{if } j = t + 2 \\ 1 & \text{if } j = 2t \\ -1 & \text{if } j = 2t + 1 \\ 0 & \text{otherwise} \end{cases}$$

Set $v^3 = \sum_{i=1}^{t-3} \Phi(v^2, i)$. For any j , $1 \leq j \leq L$,

$$v^3(j) = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = t - 2 \\ -1 & \text{if } j = t - 1 \\ -1 & \text{if } j = t + 1 \\ 1 & \text{if } j = 3t - 2 \\ 1 & \text{if } j = 3t \\ -1 & \text{if } j = 4t + 1 \\ 0 & \text{otherwise} \end{cases}$$

Set $v^4 = \sum_{i=1}^t \Phi(v^3, i)$. As v^3 has a constant number of non zero values, then for any j , $1 \leq j \leq L$, $v^4(j) \geq -c$ where $c \leq 4$.

Clearly, each vector of difference from v^0 to v^4 is c -balanced where c is constant. Thus, we get that the total number of 3-deviations is $\theta(t^4)$ and the number of nodes is $\theta(t^2)$. \square

3. GENERAL CASE: LARGEST COLLUSIONS

The uniform case we analyzed so far may be described as a “clique with enemies”: A pair of nodes may either be connected by an edge with unit weight, $w(u, v) = 1$, or they are in conflict $w(u, v) = -M$, where M is sufficiently large to ensure that people in conflict never choose the same color.

More generally, a coloring game may be defined with weights taking values in a larger set. In such, nodes choose to interact with each others according to more complex preferences.

We first show that 1-deviations in a general weighted coloring game result in an increased total utility. Hence,

THEOREM 4. *For any weighted graph, if only 1-deviations are allowed, a coloring game always converges in $O(n^2)$ steps.*

PROOF. This is clear as the global utility increases at each 1-deviation, without any concern about the values of the weights. Indeed, it only requires symmetrical relations between nodes to hold. \square

This proves that all these games admit a configuration that is 1-stable, or equivalently, a Nash-equilibrium.

However, these games are much harder to characterize than the uniform case: A k -stable configuration (with $k > 1$) may not necessarily exist, and as a result, the dynamics of the game may not always converge. It may also fail to converge even when such configurations exist. We now analyze all games where weights are chosen in a fixed subset $\mathcal{W} \subseteq \mathbb{Z}$. We wish to identify the greatest values of k for which k -stability can be attained, that we denote $k(\mathcal{W})$.

3.1 Games with weights $\mathcal{W} = \{-M, 0, 1\}$

We first focus on the set $\mathcal{W} = \{-M, 0, 1\}$, of theoretical importance. Whereas it does not differ that much from the set $\{-M, 1\}$, its stability properties are far more constrained. If we refer ourselves to the dynamic system, then there may not exist stable configurations whenever the global utility does not increase at each step. We prove that in this case $k(\mathcal{W}) = 2$.

Let us first show that $k(\mathcal{W}) \geq 2$. In fact, Theorem 5 is a global stability result, which is more precise and uses structural properties of the graphs. Let us define the “friendship graph” $G^+ = (V, E^+)$ of $G = (V, E, w)$, where $E^+ = \{\{u, v\} \in E : w(u, v) > 0\}$. We remind that the *girth* of a graph is the length of its shortest cycle. By definition, an acyclic graph has infinite girth.

THEOREM 5. *Given an integer $k \geq 1$ and a graph $G = (V, E, w)$ with $\mathcal{W} = \{-M, 0, 1\}$, there exists a k -stable configuration for G if the girth of the friendship graph G^+ is at least $k + 1$. In that case, $L(k, n) = O(n^2)$.*

PROOF. Let C_0 be any configuration for G , such that there is no vertices u and v with $w(u, v) = -M$ and $C_0(u) = C_0(v)$. If C_0 is k -stable, we are done. Otherwise there exists a subset $S = \{v_1, v_2, \dots, v_{|S|}\}$, $|S| \leq k$, such that S breaks C_0 . As in the case $k = 1$, we will prove the global utility strictly increases after any k -deviation. Let C_1 be the new configuration for G that we get after the k -deviation. Actually, we already have that, for any $1 \leq i \leq |S|$, $f_{v_i}(C_1) \geq f_{v_i}(C_0) + 1$. However, we have to be careful with summations, as we may have v_i that benefits from the presence of v_j , for some $1 \leq j \leq |S|$, $j \neq i$. Such improvement of the individual utility of both v_i and v_j cannot be counted twice for the global utility. Thus, by symmetry, we get $f(C_1) - f(C_0) \geq 2|S| - 2 \sum_{1 \leq i < j \leq |S|} w(v_i, v_j)$.

Moreover, we claim that for every $1 \leq i < j \leq |S|$, $w(v_i, v_j) \neq -M$, because M is taken large enough so that such an interaction is avoided at any cost, if it does not already exist. Consequently, either $w(v_i, v_j) = 0$, or $w(v_i, v_j) = 1$. Hence, by definition of G^+ , $f(C_1) - f(C_0) \geq 2(|S| - |E_S|)$ where E_S is the set of edges induced by nodes of S in the friendship graph G^+ . Furthermore, G^+ has girth at least $k + 1$, that is the induced graph $G_S = (S, E_S)$ is a forest, because $|S| \leq k$, and so $|E_S| < |S|$. Finally, since the global utility is upper-bounded, we get a k -stable configuration for G after a finite number of k -deviations. \square

The proof of Theorem 5 uses a potential function based on global utility. Particularly, if G^+ is cycle-free, then we get there is a k -stable configuration for G , for any $k \geq 1$; if G^+ is triangle-free, then there always exists a 3-stable configuration for G . Furthermore, as the girth of any graph is at least 3, then there always exists a 2-stable configuration ($k(\{-M, 0, 1\}) \geq 2$). It also shows that the game takes at most $O(n^2)$ steps to converge. Using another sequence construction, one can prove that this bound is indeed tight in the general case.

LEMMA 5. $\mathcal{W} = \{-M, 0, 1\}$, $L_{\mathcal{W}}(1, n) = L(2, n) = \theta(n^2)$.

PROOF. Let any $p \geq 1$. Let $G = (V, E)$ be the graph constructed as follows. Without loss of generality assume $n = 3p$. Let $V = V^1 \cup V^2 \cup V^3$ with $|V^1| = |V^2| = |V^3| = p$. For any $u, v \in V^2 \cup V^3$, $w(u, v) = 1$. For any $u, v \in V^1$, $w(u, v) = 0$. For any $u \in V^1$, for any $v \in V^2$, $w(u, v) = 1$.

We set $V^1 = \{u_1, u_2, \dots, u_p\}$ and $V^2 = \{v_1, v_2, \dots, v_p\}$.

Consider the partition C such that any node of $V^1 \cup V^2$ forms a singleton group and there is group formed by all the nodes of V^3 .

Sequentially, each node of V^1 reaches the group of v_1 . The utility of v_1 is now p . Then v_1 reaches the group composed of all nodes of V^3 . The utility of v_1 is now $p + 1$, and the utility of each u_i is 0. The number of 1-deviations is $p + 1$. We repeat the same process for v_2 using $p + 1$ deviations. And so on for each v_i , $3 \leq i \leq p$.

Finally, the number of 1-deviations is $(p + 1)p$. Thus $L_{\{-M, 0, 1\}}(1, n) = \theta(n^2)$ because the global utility, upper-bounded by $n(n - 1)$, strictly increases after each 1-deviation.

Obviously, we get $L_{\{-M, 0, 1\}}(2, n) = \theta(n^2)$ \square

Finally, we show that there exists a graph G such that any configuration for G is not 3-stable ($k(\{-M, 0, 1\}) \leq 2$).

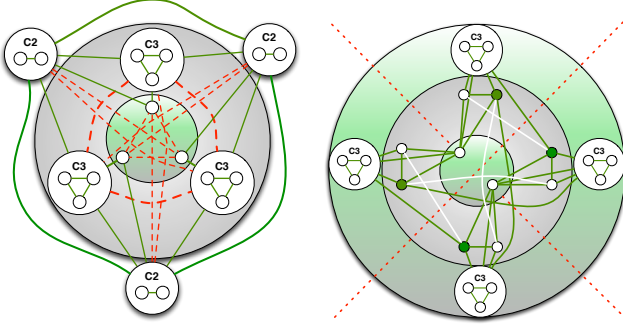


Figure 4: Graph that does not admit a k -stable configuration with $k=4$ (left) and $k=3$ (right).

The proof is all the more counter-intuitive as it implies there are infinite sequences of deviations that do not change the global utility, even in graphs with only four edges weighted by 0!

We first define the notion of twin vertices and prove a useful lemma.

DEFINITION 5. Let $G = (V, E, w)$ be a graph, and $u, v \in V$ be two vertices. We say that u and v are twins if and only if $w(u, v) > 0$ and for all vertex $s \in V$ $w(u, s) = w(v, s)$.

LEMMA 6. Given a graph $G = (V, E, w)$, and a 1-stable configuration C for G , for all vertices u, v such that u and v are twins, we have that $C(u) = C(v)$.

PROOF. By contradiction. Suppose $C(u) \neq C(v)$. By symmetry, we can assume $f_u(C) \leq f_v(C)$. Thus, $\{u\}$ breaks the configuration C , as vertex u joins the group of v because, in this new configuration C' , we have $f_u(C') = f_v(C) + w(u, v) > f_u(C)$. A contradiction. Hence $C(u) = C(v)$. \square

LEMMA 7. Let $\mathcal{W} = \{-M, 0, 1\}$. There exists a graph $G = (V, E, w)$ such that any configuration with single channel for G is not 3-stable.

PROOF. We define the graph $G = (V, E, w)$ as follows. The set of vertices V is partitioned into five subsets: A_0, A_1, A_2, A_3 and $\{b_0, b_1, b_2, b_3, c_0, c_1\}$. Let i, i', j, j' be four integers (not necessarily distinct), such that $0 \leq i, i', j, j' \leq 3$.

- First we have that $|A_i| = 4$, and we denote the vertices in A_i by $a_{i,0}, a_{i,1}, a_{i,2}$ and $a_{i,3}$.
- Furthermore, we set $w(a_{i,j}, a_{i',j'}) = 1$ if $i = i'$, and $w(a_{i,j}, a_{i',j'}) = -M$ otherwise.
- We also set $w(b_i, a_{i',j}) = 1$ if $i = i'$, or $i' \equiv i + 1 \pmod{4}$ and $j \neq i$; $w(b_i, a_{i',j}) = 0$ if $i' \equiv i + 1 \pmod{4}$ and $j = i$; $w(b_i, a_{i',j}) = -M$ otherwise.
- Moreover, $w(b_i, b_{i'}) = 1$ if $i' \equiv i \pm 1 \pmod{4}$; $w(b_i, b_{i'}) = -M$ if $i' \equiv i + 2 \pmod{4}$.
- Finally, for every $0 \leq p \leq 1$, $w(b_i, c_p) = 1$, $w(c_p, a_{i,j}) = 1$ if $i \equiv p \pmod{2}$, and $w(c_p, a_{i,j}) = -M$ otherwise.
- We also set $w(c_0, c_1) = -M$.

Observe that there are only four edges in G whose weight equals 0. Figure 4 (right) represents the graph G .

We now assume there exists a 3-stable configuration C_3 for G .

We first claim that every subset A_i is a subgroup in C_3 , that is all the vertices in A_i are in the same group in C_3 . Indeed, $A_i \setminus \{a_{i,i}\}$ is a subgroup in C_3 by Lemma 6. Then, by contradiction, suppose $a_{i,i}$ is not in the same group as $A_i \setminus \{a_{i,i}\}$. Observe that all the vertices in A_i share the same enemies. Moreover, the only friends of $a_{i,i}$ that are not in A_i are b_i and $c_{i \pmod{2}}$. Thus $f_{a_{i,i}}(C_3) \leq 2$, so, $\{a_{i,i}\}$ breaks C_3 , and the claim is proved. At the end, there are, at least, four groups in C_3 (one different group for each subset A_i).

We now claim that either b_i is in the same group as A_i in C_3 , or b_i is in the same group as $A_{i'}$ in C_3 , with $i' \equiv i + 1 \pmod{4}$. By contradiction. One can check that, if b_i is neither in the same group as A_i nor in the same group as $A_{i'}$, we have $f_{b_i}(C_3) \leq 2$, due to the conflict graph in G . On the opposite, if either b_i is in the same group as A_i in C_3 , or b_i is in the same group as $A_{i'}$ in C_3 , we get $f_{b_i}(C_3) \geq 3$, and there is no enemy of b_i that is not enemy with the vertices in A_i or that is not enemy with the vertices in $A_{i'}$. Again, there is a contradiction, as in this case, C_3 would not be 1-stable.

Especially, we claim that there is an i , $0 \leq i \leq 3$, such that b_i and $b_{i''}$ are in the same group as A_i in C_3 , with $i \equiv i'' + 1 \pmod{4}$. To prove this claim, we first recall that $w(b_i, b_{i''}) = 1$. Then, again, we prove this claim by contradiction. Suppose that, for every $0 \leq i \leq 3$, there is only one vertex b_j such that b_j and A_i are in the same group. By the hypothesis, either $j = i$, or $j = i''$. Furthermore, either $j = i$ for all i , or $j = i''$ for all i . We also have that either c_0 is in the same group as A_0 , or c_0 is in the same group as A_2 . In the same way, we have that either c_1 is in the same group as A_1 , or c_1 is in the same group as A_3 . Moreover, for every $0 \leq i \leq 3$, $w(c_0, b_i) = w(c_1, b_i) = 1$. Thus, if $j = i''$ for all i , then any subset $\{b_{i''}, c_{i'' \pmod{2}}\}$ breaks the configuration C_3 , by joining the same group as $A_{i''}$, and so, $j = i$ for all i , because C_3 is 3-stable. We have $f_{b_i}(C_3) = 4$ if $c_{i \pmod{2}}$ is not in the same group as b_i and A_i , and $f_{b_i}(C_3) = 5$ otherwise. However, remark that there are always two vertices b_i such that $f_{b_i}(C_3) = 4$, because there are only two vertices c_0, c_1 . Furthermore, if $f_{b_i}(C_3) = 4$, then the subset $\{b_i, c_{i' \pmod{2}}\}$ breaks the configuration C_3 , by joining the same group as $A_{i'}$, with $i' \equiv i + 1 \pmod{4}$. Hence C_3 cannot be 2-stable, which is a contradiction, and our third claim is proved.

As an immediate consequence, we get that there exists an i , $0 \leq i \leq 3$, such that $b_i, b_{i''}$ and $c_{i \pmod{2}}$ are in the same group as A_i in C , with $i \equiv i'' + 1 \pmod{4}$. Such a group is unique in C_3 , due to the conflict graph in G . By symmetry, suppose that $i = 0$.

There are four cases.

1. If b_1 and b_2 are in the same group as A_2 in C_3 , then $\{b_1, c_1\}$ breaks the configuration C_3 , by joining the same group as A_1 .
2. If b_2 is in the same group as A_3 , and b_1 and c_1 are in the same group as A_1 , then $\{b_2\}$ breaks the configuration C_3 , by joining the same group as A_2 .
3. If b_2 and c_1 are in the same group as A_3 , then $\{b_3\}$ breaks the configuration C_3 , by joining the same group as A_3 .

4. If b_2 is in the same group as A_2 , and b_1 and c_1 are in the same group as A_1 , then $\{b_2, b_3, c_1\}$ breaks the configuration C_3 , by joining the same group as A_3 .

Finally, there does not exist a 3-stable configuration C_3 for G . \square

Figure 4 presents two graphs that are not stable for $k = 4$ and $k = 3$, respectively. To keep graphs readable, we use conventions. (1) Some cliques of nodes are grouped within a circle (e.g., C_3 is a clique of 3 nodes); an edge from another node to that circle denotes an edge to *all* elements of this clique. (2) On the right, we omitted the edges with weights $-M$ denoting pairs of enemies. These are all the edges between nodes on different sides of a dashed line, unless it was otherwise indicated as a green edge (weight 1) or a white edge (weight 0). We deduce then:

LEMMA 8. *Let $\mathcal{W} = \{-M, 0, 1\}$. Then $k(\mathcal{W}) = 2$.*

3.2 Game with general weights

While the two results we obtained ($k \geq 1$ in general, $k = 2$ when the set of weights contains $-M, 0$ and 1) seems constrained, we now prove that these are the best results that one can hope for.

First, we observe that all results from the previous section hold when more negative weights are added to the set \mathcal{W} , hence we have $k(-\mathbb{N} \cup \{-M, 0, 1\}) = 2$. However, as we show through a counter-example, no 2-stable configuration exists in general when \mathcal{W} contains two positive elements, even when the zero is not present.

LEMMA 9. *Given two positive integers a, b such that $a < b$, there is a graph $G = (V, E, w)$ with $\mathcal{W} = \{-M, a, b\}$ such that there is no 2-stable configuration for G .*

PROOF. We build a graph $G = (V, E, w)$ as follows. The set V can be partitioned into four sets $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ and $\{u_1, u_2, u_3\}$, such that all the edges between two vertices of the same subset have weight b . Moreover, all the edges that lie between A and B , B and C , and C and A have weight $-M$. We also set that for all vertices $a_i \in A$, for all vertices $b_j \in B$, and for all vertices $c_l \in C$, $w(a_i, u_1) = w(b_j, u_2) = w(c_l, u_3) = b$, whereas $w(a_i, u_3) = w(b_j, u_1) = w(c_l, u_2) = -M$. Finally:

- $w(u_1, c_1) = w(u_1, c_2) = b$; $w(u_1, c_3) = a$;
- $w(u_2, a_1) = w(u_2, a_2) = b$; $w(u_2, a_3) = a$;
- $w(u_3, b_1) = w(u_3, b_2) = b$; $w(u_3, b_3) = a$.

Assume there exists a 2-stable configuration C_2 for G .

We claim that A, B and C are subgroups in C_2 . Indeed, it is clear by Lemma 6 that $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ are subgroups in C . By symmetry, let us only consider the case of a_3 . Remark that all the vertices in A share the same enemies in the graph. Moreover, the sum of the positive weights that lie between u_3 and $V \setminus A$ equals $b + a$, which is lower than $2b$. Hence A is a subgroup of C_2 because C_2 is 1-stable.

We now claim that, for any $u_i \in \{u_1, u_2, u_3\}$, there exists a subgroup $K \in \{A, B, C\}$ such that $K \subset C_2(u_i)$. By symmetry, we only need to show it for u_1 . This is clear as $w(u_1, u_2) + w(u_1, u_3) = 2b < 2b + a$, and the enemies of u_1 are contained within the enemies of A and C .

At last, we have either $A \cup \{u_1, u_2\}$ or $B \cup \{u_2, u_3\}$ or $C \cup \{u_3, u_1\}$ which is a group in C_2 . By symmetry, we can suppose $A \cup \{u_1, u_2\}$ is a group in C_2 . Then, we have $\{u_2, u_3\}$ that breaks C_2 , and so C_2 cannot be 2-stable. A contradiction.

As a consequence, there does not exist any 2-stable configuration for G . \square

COROLLARY 2. *Let \mathcal{W} be a set of weights, such that $\{-M, N\} \subset \mathcal{W}$.³ There exists a graph $G = (V, E, w)$ such that any configuration for G is not 3-stable.*

PROOF. By Lemma 10, there exists a graph $G_0 = (V_0, E_0, w_0)$ with $\{-M, 0, 1\}$ such that there does not exist any 3-stable configuration for G_0 . Now, let $a \in \mathcal{W} \setminus \{-M, N\}$. Note that a may be positive or non positive. We define $G_1 = (V_0, E_0, w_1)$ such that, for all vertices $u, v \in V_0$, $w_1(u, v) = -M$ if $w_0(u, v) = -M$, $w_1(u, v) = N$ if $w_0(u, v) = 1$, $w_1(u, v) = a$ if $w_0(u, v) = 0$. As there does not exist any 3-stable configuration for G_0 , there does not exist any 3-stable configuration for G_1 either. \square

LEMMA 10. *Given two positive integers a, b , let $\mathcal{W} = \{-a, b\}$. There exists a graph $G = (V, E, w)$ such that any configuration for G is not $2(b + a + 1)$ -stable.*

PROOF. We emulate the triangle of Lemma 9, this time with the three weights $b + a + 1$, $2(b + a) + 3$, $3(b + a + 1)$, instead of 2, 3 and 4 respectively. As $b + a + 1 < 2(b + a) + 3 < 3(b + a + 1)$, and $b + a + 1 + 2(b + a) + 3 = 3(b + a + 1) + 1 > 3(b + a + 1)$, this change of weights does not make 2-stable configurations exist for the counter-example of Lemma 9. However, we also have to emulate a weight 0. To do that, let t_1, t_2, t_3 be any three positive integers that satisfy the following properties:

- $t_3 \geq t_2 \geq t_1$;
- $t_1 \geq 5(b + a + 1) / \max(b - a, 1)$;
- $\max(t_1 b, (t_1 + 1)a + 2(b + a) + 3) \cdot a \geq (3(b + a) + 2)b + 1$;
- $\max(t_2 b, (t_2 + 1)a + 2(b + a) + 3) \cdot a \geq [(t_1 + 5)(b + a) + 4]b + 1$;
- $\max(t_3 b, (t_3 + 1)a + 2(b + a) + 3) \cdot a \geq [(t_2 + 5)(b + a) + 4]b + 1$.

We can note the possible values of those three integers are infinite. Indeed, the value of t_1 can be any value that is greater than every of its (finite) lower bounds. Furthermore, if the value of t_1 is arbitrarily fixed, then we can also choose for t_2 any value that is sufficiently large and, finally, the value of t_3 can be chosen in the same way, depending on the values of t_1 and t_2 . For every $1 \leq i \leq 3$, we also define the size $s_i = t_i(b + a) + 3(b + a + 1) = (t_i + 1)(b + a) + 2(b + a) + 3$.

We now define the graph $G = (V, E, w)$ such that $V = V_1 \cup V_2 \cup V_3 \cup A_1 \cup A_2 \cup A_3$, $|A_1| = |A_2| = |A_3| = b + a + 1$, and for any $1 \leq i \leq 3$, $|V_i| = s_i$, as follows.

For every $1 \leq i \leq 3$, there are two distinct subcliques $V_{i,m} \subset V_{i,p} \subset V_i$, with $|V_{i,m}| = t_i b$, $|V_{i,p}| = (t_i + 1)b$; furthermore, for all $v, v' \in V_i$, $w(v, v') = b$.

We also have that $w(u, v) = b$ for any $u, v \in A_1 \cup A_2 \cup A_3$.

We set:

³A weight N denotes a best friend if it is far larger (in absolute value) than any weight in $\mathcal{W} \setminus \{-M\}$.

- $w(u, v) = b$, for any $u \in A_1$, any $v \in V_1 \setminus V_{1,m}$,
- $w(u, v) = b$, for any $u \in A_2$, any $v \in V_2 \setminus V_{2,m}$,
- $w(u, v) = b$, for any $u \in A_3$, any $v \in V_3 \setminus V_{3,m}$.

Moreover,

- $w(u, v) = b$, for any $u \in A_1$, any $v \in V_2 \setminus V_{2,p}$,
- $w(u, v) = b$, for any $u \in A_2$, any $v \in V_3 \setminus V_{3,p}$,
- $w(u, v) = b$, for any $u \in A_3$, any $v \in V_1 \setminus V_{1,p}$.

Every other weight equals $-a$.

Let C be any stable configuration for G . By Lemma 6, we get subsets $A_1, A_2, A_3, V_{1,m}, V_{2,m}, V_{3,m}, V_1 \setminus V_{1,p}, V_2 \setminus V_{2,p}, V_3 \setminus V_{3,p}$ are subgroups in C .

CLAIM 8. *For every $1 \leq i \leq 3$, either $a \geq b$ and $|V_{i,p}| + 2(b+a+1) \leq |V_i \setminus V_{i,p}|$, or $b > a$ and $|V_i \setminus V_{i,m}| + 2(b+a+1) \leq |V_{i,m}|$.*

By the hypothesis, we have $|V_{i,p}| = (t_i + 1)b$, hence $|V_i \setminus V_{i,p}| = s_i - (t_i + 1)b = (t_i + 1)(b+a) + 2(b+a) + 3 - (t_i + 1)b = (t_i + 1)a + 2(b+a) + 3$; furthermore, $|V_{i,m}| = t_i b$, and so, $|V_i \setminus V_{i,m}| = s_i - t_i b = t_i(b+a) + 3(b+a+1) - t_i b = t_i a + 3(b+a+1)$.

First if $a \geq b$, then $|V_{i,p}| + 2(b+a+1) = (t_i + 1)b + 2(b+a+1) < (t_i + 1)a + 2(b+a) + 3 = |V_i \setminus V_{i,p}|$, and the claim is proved. Otherwise, the equation $|V_i \setminus V_{i,m}| + 2(b+a+1) \leq |V_{i,m}|$ is equivalent to the following lines:

$$t_i a + 3(b+a+1) + 2(b+a+1) \leq t_i b$$

$$5(b+a+1) \leq t_i(b-a)$$

$$5(b+a+1)/(b-a) \leq t_i$$

and the claim is also proved, as $t_i \geq t_1 \geq 5(b+a+1)/\max(b-a, 1)$.

In the sequel, we denote $V'_i = V_i \setminus V_{i,p}$ if $a \geq b$, and $V'_i = V_{i,m}$ if $b > a$. We recall that V'_i is a subgroup in C .

CLAIM 9. *For any $j < i$, none of the vertices in V_j is in the same group as vertices of V'_i .*

Any vertex in V_j has at most $s_j - 1 + 2(b+a+1) = (t_j + 1)(b+a) + 2(b+a) + 3 + 2(b+a+1) - 1 = (t_j + 5)(b+a) + 4$ friends in its group. Moreover, $[(t_j + 5)(b+a) + 4]b < \max(t_i b, (t_i + 1)a + 2(b+a) + 3) \cdot a = |V'_i|a$ by the hypothesis. So, the second claim is proved.

CLAIM 10. *Subsets A_1 and V'_3 are not in the same group in C .*

By contradiction. Suppose $A_1 \cup V'_3$ is a subgroup in C . By the previous claim, there can be no vertex of $V_1 \cup V_2$ in the same group as A_1 . Thus, any vertex of A_1 has at most $2(b+a+1) + b+a = 3(b+a) + 2$ friends in its group. Furthermore, as $|V'_3|a = \max(t_3 b, (t_3 + 1)a + 2(b+a) + 3) \cdot a \geq (3(b+a) + 2)b + 1$, we have that any vertex of A_1 has a negative utility, which is a contradiction. So, the third claim is proved.

In the same way, A_2 and V_1 are not in the same group in C , A_3 and V_2 are not in the same group in C .

$k(\mathcal{W})$	\mathcal{W}
1	$\{-M, a, b\}, 0 < a < b$
2	$\{-M, -\mathbb{N}, 0, 1\}$
∞	$\{-M, b\}, b > 0; \mathcal{W} \subseteq \mathbb{N}; \mathcal{W} \subseteq -\mathbb{N}; \mathcal{W} = -\mathbb{N} \cup \{N\};$

Table 2: Values of $k(\mathcal{W})$ for different \mathcal{W} .

CLAIM 11. *For any $1 \leq i \leq 3$, V_i is a subgroup in C .*

By our first claim, $|V_3 \setminus V'_3| + 2(b+a+1) \leq |V_3|'$, and by our third claim, there can be no vertex u in the same group as V'_3 such that $w(u, v_3) = -a$ for some vertex $v_3 \in V_3$. Consequently, V_3 is a subgroup in C , as any vertex of $V_3 \setminus V'_3$ that is not in the same group as C breaks the configuration. It follows that V_2 is a subgroup in C , hence V_1 is a subgroup in C .

CLAIM 12. *Given an i , $1 \leq i \leq 3$, if A_i and V_i are in the same group in C , then for any vertex $u_i \in A_i$, we get $f_C(u_i) \geq 3(b+a+1)b + (b+a)b$.*

It follows from $(|A_i| - 1)b + |V_i \setminus V_{i,m}|b - |V_{i,m}|a = (b+a)b + (t_i a + 3(b+a+1))b - t_i b a = 3(b+a+1)b + (b+a)b$.

In the same way, if A_1 and V_2 are in the same group, then for any vertex $u_1 \in A_1$, $f_C(u_1) \geq (|A_1| - 1)b + |V_2 \setminus V_{2,p}|b - |V_{2,p}|a = (b+a)b + ((t_2 + 1)a + 2(b+a) + 3)b - (t_2 + 1)b = (2(b+a) + 3)b + (b+a)b$; if A_2 and V_3 are in the same group, then for any vertex $u_2 \in A_2$, $f_C(u_2) \geq (2(b+a) + 3)b + (b+a)b$; if A_3 and V_1 are in the same group, then for any vertex $u_3 \in A_3$, $f_C(u_3) \geq (2(b+a) + 3)b + (b+a)b$.

Finally, for every stable configuration C for G , there is a corresponding configuration C' for the counter-example of Lemma 9; moreover, we can mimic any 2-deviation that breaks C' by a $2(b+a+1)$ -deviation that breaks C , and so, there does not exist any $2(b+a+1)$ -stable configuration for G . \square

Table 2 summarizes the most important tight values for $k(\mathcal{W})$. Note in particular that we prove $k(\mathcal{W}) = \infty$ if and only if we have one of the following cases: $\mathcal{W} = \{-M, b\}$ which is equivalent to the uniform case, \mathcal{W} has no negative (or no positive) elements, in which case the game is trivial, and \mathcal{W} has only one very large weight N larger than any negative ones, in which case the stable configuration derives from connected components with edges of weight N .

More formally, let us decompose any set of weights \mathcal{W} into the set of non negative weights \mathcal{W}^+ and into the set of non positive weights \mathcal{W}^- .

LEMMA 11. *$k(\mathcal{W}) = \infty$ if, and only if, $\mathcal{W} = \mathcal{W}^+$, or $\mathcal{W} = \mathcal{W}^-$, or $\mathcal{W} = \{-M, b\}$, or $\mathcal{W} = \mathcal{W}^- \cup \{N\}$ and $-M \notin \mathcal{W}$.*

PROOF. First note that any other \mathcal{W} is such that $k(\mathcal{W})$ is upper-bounded, by the lemmas that are above in this section. We start the proof with a preliminary remark. Let $G = (V, E, w)$ be any graph, and let C be any configuration for G . For any node $u \in V$, $f_u(C) \geq \sum_{v \in V} \min(w(u, v), 0)$ and $f_u(C) \leq \sum_{v \in V} \max(w(u, v), 0)$.

Let $\mathcal{W} = \mathcal{W}^+$. By the first remark, $f_u(C) \leq \sum_{v \in V} \max(w(u, v), 0)$ for any node $u \in V$ and for any configuration C for G . In our case, $\sum_{v \in V} \max(w(u, v), 0) = \sum_{v \in V} w(u, v)$ because $w(u, v) \geq 0$ for any $(u, v) \in E$. Let $C = (V)$ be the configuration for G composed of 1 group of

size $|V|$. For any node $u \in V$, $f_u(C) = \sum_{v \in V} w(u, v)$. Thus C is a k -stable configuration for G .

Let $\mathcal{W} = \mathcal{W}^-$. By the very first remark, $f_u(C) \leq \sum_{v \in V} \max(w(u, v), 0)$ for any node $u \in V$ and for any configuration C for G . In our case, $\sum_{v \in V} \max(w(u, v), 0) = \sum_{v \in V} 0 = 0$ because $w(u, v) \leq 0$ for any $(u, v) \in E$. Let $V = \{u_1, \dots, u_n\}$. By assumption, for any node $u \in V$ and for any configuration C for G , $f_u(C) \leq 0$ because $w(u, v) \leq 0$ for any $v \in V$. Let $C = (\{u_1\}, \dots, \{u_n\})$ be the configuration for G composed of $|V|$ groups of size 1. C is a k -stable configuration for G because $f_u(C) = 0$ for any $u \in V$.

The case $\mathcal{W} = \{-M, b\}$ is proved in [9].

Let $\mathcal{W} = \mathcal{W}^- \cup \{N\}$ such that $-M \notin \mathcal{W}$. Let $G' = (V, E')$ be the graph induced by the set of arcs $E' = \{(u, v), w(u, v) = N, (u, v) \in E\}$. Let G_1, \dots, G_t be the different connected components of G' . Given a node $u \in V$, let $\Gamma_{G'}(u)$ be the set of neighbors of u in graph G' . Formally $\Gamma_{G'}(u) = \{v, (u, v) \in E'\}$. By the preliminary remark, we get that, for any configuration C for G and for any node $u \in V$, $f_u(C) \leq N|\Gamma_{G'}(u)|$.

Let $C = (V(G_1), \dots, V(G_t))$ be the configuration for G composed of the nodes of the different maximal connected components of G . By Definition of N , $f_u(C) \geq \max((N - 1), 0)|\Gamma_G(u)|$.

We now prove that C is k -stable. By contradiction. Suppose that there exists a k -deviation. Let C' be the configuration for G obtained after the k -deviation. There are two cases.

- A subset $S \subset V(G_i)$, for some i , $1 \leq i \leq t$, $|S| \leq k$, forms a new group. In that case, we prove that at least one node $u \in S$ is such that $f_{C'}(u) < f_u(C)$. Indeed, consider a node $u \in S$ such that $\Gamma_{G'}(u) \cap S \neq \Gamma_G(u)$. Note that there always exists such a node because $|S| < |V(G_i)|$ and $V(G_i)$ is a maximal connected component of G . Thus, we get $f_{C'}(u) \leq \max((N - 1), 0)|\Gamma_{G'}(u)|$ while $f_u(C) \geq \max((N - 1), 0)|\Gamma_G(u)|$. So $f_{C'}(u) \leq f_u(C)$. A contradiction.
- A subset $S \subset V$, $|S| \leq k$, reaches a group $V(G_j)$, for some i , $1 \leq i \leq t$. If $V(G_i) \cap S \neq \emptyset$, then $V(G_i) \subset S$. Otherwise, we would have $f_{C'}(u) \leq f_u(C)$ for some $u \in V(G_i)$. Indeed, consider a node $u \in S$ such that $\Gamma_{G'}(u) \cap S \neq \Gamma_G(u)$. So $f_{C'}(u) \leq \max((N - 1), 0)|\Gamma_{G'}(u)|$ while $f_u(C) \geq \max((N - 1), 0)|\Gamma_G(u)|$. Thus, if $V(G_i) \cap S \neq \emptyset$, then $V(G_i) \subset S$. Consider a set $V(G_i)$, for some i , such that $V(G_i) \subset S$. In that case, $f_u(C) \leq f_{C'}(u)$ for any $u \in V(G_i)$ because $w(u, v) \leq 0$ for any $v \in V \setminus V(G_i)$. A contradiction.

There does not exist a k -deviation for C and so C is k -stable. \square

3.3 Intractability with conflict graphs

Under general weights, we proved that it may not be easy to ensure that all games have a k -stable configuration. We now prove a much stronger result: not only is the stability not guaranteed for a given game when $k > k(\mathcal{W})$, but it is computationally prohibitive to decide it. We first define:

DEFINITION 6 (K-STABLE DECISION PROBLEM). *Let $k \geq 1$ and let \mathcal{W} be the set of weights. Given a graph $G = (V, E, w)$, does there exist a k -stable configuration?*

The following Theorem shows that $k(\mathcal{W})$ determines the complexity of the K-STABLE DECISION PROBLEM.

THEOREM 6. *For $k \geq 1$ and \mathcal{W} containing $-M$, either*

- *a k -stable configuration always exists (i.e., $k \leq k(\mathcal{W})$);*
- *or the K-STABLE DECISION PROBLEM is NP-hard.*

A previous result of NP-hardness was used in [9]. However, their proof requires gossip deviations and a large positive weight N , whereas ours overrules these strong constraints. What we first need to show is that there are instances for our decision problem that are equivalent. Given a set of weights \mathcal{W} such that $\mathcal{W}^+ \neq \emptyset$, $w_p \in \mathcal{W}$ is defined as the largest positive weight in \mathcal{W} .

Then we present two different ways to increase the minimum utility of the nodes in a graph. In the general case, that minimum utility equals 0. However, we can improve it with external cliques, in a way that keeps the properties of the graph safe. Given a graph $G = (V, E, w)$ and a positive integer t , we build the graph \tilde{G}_t by adding to the set of vertices n cliques of t nodes. We further impose that all the nodes in the same t -clique have weight between them and between the vertices in V equal to w_p , whereas they have a conflict edge with any other vertex in the remaining t -cliques. In so doing, we intuitively increase the minimal utility of the nodes to $w_p t$. Note that the transformation also keeps the stability properties of the graph safe, which can be formalized as follows:

DEFINITION 7. *Let \mathcal{W} be any set of weights such that $\mathcal{W}^+ \neq \emptyset$. Given a graph $G = (V, E, w)$ such that $n = |V|$, and a positive integer t , for every $1 \leq i \leq n$ we can define the clique graph $K_t^i = (V_i, E_i, w_i)$ with t nodes and the set of weights $\{w_p\}$. The graph $\tilde{G}_t = (\tilde{V}, \tilde{E}, \tilde{w})$ is then built as follows:*

- $\tilde{V} = V \cup \bigcup_{i=1}^n V_i$;
- *for every $1 \leq i, j \leq n$, for all $u_i \in V_i$, and for all $u_j \in V_j$, $\tilde{w}(u_i, u_j) = w_i(u_i, u_j) = w_p$ if $i = j$, $\tilde{w}(u_i, u_j) = -M$ otherwise;*
- *for all $u, v \in V$, $\tilde{w}(u, v) = w(u, v)$;*
- *for all $1 \leq i \leq n$, for all $u_i \in V_i$ and for all vertex $v \in V$, $\tilde{w}(u_i, v) = w_p$.*

Another way is to substitute every vertex in the graph by cliques. By this second process, we get the graph \tilde{G}_α , and it keeps the stability properties of G . Formally:

DEFINITION 8. *Let \mathcal{W} be any set of weights such that $\mathcal{W}^+ \neq \emptyset$. Given a graph $G = (V, E, w)$ such that $n = |V|$, and a positive integer α , for every vertex $u \in V$ we can define the clique graph $K_u = (V_u, E_u, w_u)$ with α nodes and the set of weights $\{w_p\}$. The graph $\tilde{G}_\alpha = (\tilde{V}_\alpha, \tilde{E}_\alpha, \tilde{w}_\alpha)$ is then built as follows:*

- $\tilde{V} = \bigcup_{u \in V} V_u$;
- *for every $u \in V$, for all $u_1, u_2 \in V_u$, $\tilde{w}_\alpha(u_1, u_2) = w_u(u_1, u_2) = w_p$;*
- *for all $u, v \in V$, for all $u_i \in V_u$ and for all $v_j \in V_v$, $\tilde{w}_\alpha(u_i, v_j) = w(u, v)$.*

We still have to prove our first transformation keeps the stability properties of the graph safe.

LEMMA 12. *Let \mathcal{W} be any set of weights such that there are positive weights in \mathcal{W} , and k be a positive integer. Given a graph $G = (V, E, w)$, and a positive integer t such that $t > |V|$, there exists a k -stable configuration for G if and only if there exists a k -stable configuration for \tilde{G}_t .*

PROOF. Let $C_k = \{V_1, \dots, V_n\}$ be a k -stable configuration for G . We claim that $C'_k = \{V_1 \cup V(K_t^1), \dots, V_n \cup V(K_t^n)\}$ is a k -stable configuration for \tilde{G}_t . By contradiction. Assume that C'_k is not k -stable, and let S be a k -deviation that breaks C'_k . First, we claim that for every $1 \leq i \leq n$ S cannot be a subset of $V(K_t^i)$. This is clear as all the vertices in K_t^i have enemies in every non-empty group of C'_k that is not their own group. Thus they cannot break the configuration alone, since their utility in C'_k is positive and they are already in the same group. Then it follows that $S' = S \cap V(G) \neq \emptyset$. Moreover, we claim that S' breaks C_k . Indeed, after the k -deviation, the vertices in S' end in the same group as a (possibly empty) subset of K_t^i for some $1 \leq i \leq n$. Let $1 \leq j \leq n$ such that $V_j \cap (V(G) \setminus S') = \emptyset$. Such an integer j exists, because there are at most n non-empty groups in C_k . Then it follows that if we move S' into the group V_j it is still a valid k -deviation for C'_k , and so, we can mimic this k -deviation in C_k . A contradiction. Therefore C'_k is k -stable.

Conversely, assume that there is no k -stable configuration for G , whereas there exists a k -stable configuration $C'_k = \{V'_1, \dots, V'_n\}$ for \tilde{G}_t . Particularly, for all $1 \leq i \leq n$, we get that K_t^i is a subgroup of C'_k by Lemma 6. Furthermore, we claim that for all vertex $u \in V$, there exists an integer $i \in \{1, \dots, n\}$ such that $K_t^i \subset C_k(u)$. This comes from the fact that $t > n$ and that w_p is the largest positive weight in \mathcal{W} . Let $C_k = \{V'_1 \cap V(G), \dots, V'_n \cap V(G)\}$ be the underlying configuration for G that one can deduce from C'_k . By the hypothesis, there exists a k -deviation S that breaks C_k . However, we can mimic this deviation in C'_k , which is a contradiction because C'_k is k -stable. Consequently, there does not exist any k -stable configuration for \tilde{G}_t . \square

We are now able to prove Theorem 6:

PROOF. Suppose there exists a graph $G_0 = (V_0, E_0, w_0)$ with the set of weights \mathcal{W} , such that there does not exist any k -stable configuration for G_0 . We get there are positive weights, because otherwise the configuration with only singleton groups is k -stable for G_0 . Moreover, we can assume that there exists a node $x_0 \in V_0$ such that the removal of x_0 makes the existence of a k -stable configuration for the gotten subgraph. Indeed, otherwise, we remove nodes sequentially until obtaining this property.

Let us choose C_k a k -stable configuration for $G_0 \setminus x_0$. By the hypothesis, the configuration $C_k \cup \{x_0\}$ is not k -stable for G_0 . Let us define f_0 as the maximum utility the vertex x_0 can get with only one k -deviation that breaks $C_k \cup \{x_0\}$. We can always assume $f_0 > 0$ by using the transformation of Definition 7 if needed. We define two other constants, namely $\alpha = \lceil \frac{f_0}{w_p} \rceil$ and $c_0 = 2n_0 + 1$, with $n_0 = |V_0|$.

We can now prove the NP-hardness by using a polynomial reduction for the MAXIMUM INDEPENDENT SET PROBLEM. Let $G = (V, E)$ be a graph, and let $c \geq c_0$ be an integer. We

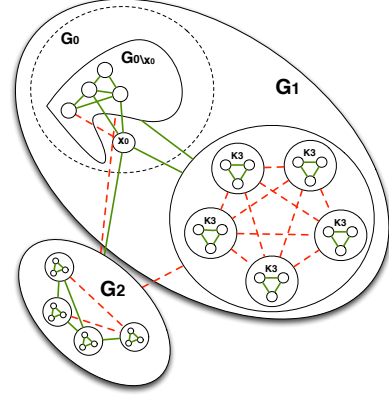


Figure 5: The transformation of an input.

identify G to a graph $D_G = (V, E, w)$, such that for all $u, v \in V$, $w(u, v) = -M$ if $uv \in E$, and $w(u, v) = w_p$ otherwise. Let t be the integer $\lfloor \alpha c - \frac{f_0}{w_p} \rfloor$. We can consider the graphs $G_1 = \tilde{G}_{0t} = (V_1, E_1, w_1)$ and $G_2 = \tilde{G}_\alpha = (V_2, E_2, w_2)$. Observe that $t > \alpha c - \frac{f_0}{w_p} - 1 \geq \alpha c - n_0 - 1 \geq c - n_0 - 1 \geq n_0$. So we can apply Lemma 12 to G_1 .

We build the graph $H_G = (V_H, E_H, w_H)$ as follows:

- $V_H = V_1 \cup V_2$;
- for all $1 \leq i \leq 2$, for all $u_i, v_i \in V_i$, $w_H(u_i, v_i) = w_i(u_i, v_i)$;
- for all $u_1 \in V_1$, for all $v_2 \in V_2$, $w_H(u_1, v_2) = w_p$ if $u_1 = x_0$, and $w_H(u_1, v_2) = -M$ otherwise.

The transformation above is illustrated in Figure 5. First assume that any independent set of G has a size lower than c . Then, there can be no group in a stable configuration with more than $\alpha(c-1)$ vertices of V_2 . Furthermore, $\alpha(c-1) = \alpha c - \lceil \frac{f_0}{w_p} \rceil - 1 \leq \alpha c - \frac{f_0}{w_p} - 1 < t$, and so, the minimum utility of x_0 in G_1 is greater than its maximum utility in G_2 . As a consequence, C_1 can be partitioned into a 1-stable configuration for G_1 and a 1-stable configuration for G_2 . Moreover, there is no k -stable configuration for G_0 by the hypothesis, hence there is no k -stable configuration for G_1 either by Lemma 12. Therefore, there is no k -stable configuration for H_G .

Conversely, assume that there exists an independent set of G with size at least c . By [9], there exists a k -stable configuration C_k^α for $G_2 \cup \{x_0\}$, such that there is a group $A_c \in C_k^\alpha$ that is a maximum independent set of the conflict graph of $G_2 \cup \{x_0\}$. Furthermore, $x_0 \in A_c$, because x_0 has no conflict within the graph. Observe that $G_1 \setminus x_0$ is $(G_0 \setminus x_0)_t$ with an extra clique graph $K_t^{n_0}$ added. Hence, both these graphs have the same stable properties. As a consequence, if $C_k = \{B_1, \dots, B_{n_0-1}\}$, then there exists a k -stable configuration C_k^t for $G_1 \setminus x_0$ by Lemma 12 that is isomorphic to C_k . We then claim that $C_k^H = C_k^\alpha \cup C_k^t$ is a k -stable configuration for H_G . Indeed, we have that the utility of x_0 in C_k^α is at least $w_p \alpha c$. Moreover, the maximum utility x_0 can get after a k -deviation that breaks C_k^t is $f_0 + w_p t = w_p(t + \frac{f_0}{w_p}) \leq w_p \alpha c$.

We can conclude the NP-hardness, as our transformation is polynomial, and the MAXIMUM INDEPENDENT SET PROBLEM is NP-complete [8]. \square

4. EXTENSIONS OF COLORING GAMES

All theoretical model of social dynamics should consider whether the overall behavior of the model is not too limited by some simplifying assumptions. We prove this is not the case, as our results extend to account for various situations in the formation of social groups.

4.1 Asymmetry and Gossiping

Studying directed graphs rather than undirected graphs is a natural generalization. In this case, we may not have that $w(u, v) = w(v, u)$ for all nodes u, v . However, even if modest generalization of the model, asymmetrical weights in the preference matrix leads to intractability. This can be seen with a simple digraph $D = (\{u, v\}, E, w)$ such that $w(u, v) > 0$ whereas $w(v, u) < 0$. Furthermore, we prove in Theorem 7 that it is NP-hard to decide whether there exists a 1-stable configuration. This result holds even when the maximum number of groups is constant. We use in our reduction the multi-way partitioning number problem, which is NP-complete [10].

THEOREM 7. *For any $k \geq 1$, deciding whether there exists a k -stable configuration for a given digraph $D = (V, A, w)$ is NP-hard, even if the number of groups is upper-bounded by any constant $g \geq 2$.*

PROOF. We constrain ourselves to $g = 2$ in order to ease the proof. This is enough so that we give the intuition of the general case.

Consider an instance S of Partition problem. Let us call $T = \sum_{s \in S} s$. We define the digraph $D = (V, A, w)$ as follows. The set of vertices is $V = S \cup \{z, u, v\}$. For all $s_1, s_2 \in S$ we set $w(s_1, s_2) = -s_2$, $w(s_1, u) = w(s_1, v) = T - s_1 + 1$, $w(u, s_1) = w(v, s_1) = 0$, and finally $w(s_1, z) = -w(z, s_1) = s_1$. We also set $w(u, v) = w(v, u) = -M$, that $w(u, z) = w(v, z) = 0$, and that $w(z, u) = w(z, v) = T + 1$. Such a digraph D can be built in polynomial time.

First, in any stable configuration for D , we get that u and v must be in two different groups because they are enemies. Furthermore, for all vertex $x \notin \{u, v\}$, $w(u, x) = w(v, x) = 0$, and so the other vertices do not matter for u and v . Hence for any configuration C such that u and v are already in two different groups, and for any k , there is no subset that contains u or v and that may break the configuration.

Moreover, we recall that $w(z, s) < 0$ for any $s \in S$. Consequently, in any configuration C for D such that z is neither in the same group as u , nor z is in the same group as v , we get that $f_z(C) \leq 0$, whereas in any configuration C' such that z is in the same group as u or z is in the same group as v , we get that $f_z(C') \geq 1$ because $\sum_{s \in S} w(z, s) = -T$ and $w(z, u) = w(z, v) = T + 1$.

Hence, for any stable configuration for D , we conclude that either z is in the same group V_0 as u , or z is in the same group V_1 as v . Without any loss of generality, suppose z in the same group V_0 as u . We claim that any vertex s in S must be also in the same group V_0 as u and z or in the same group V_1 as v . Indeed, otherwise we would have that $f_s(C) \leq 0$, since $w(s, s') \leq 0$ for any $s' \in S$; however such a configuration C is not 1-stable, because if vertex s

joins the group V_1 , then $f_s(C') > 0$, with C' being the new configuration, due to the fact that $w(s, v) = T - s + 1 > 0$, and $T - s = -[\sum_{s' \in S} w(s, s')]$. Thus, there can be only two groups in a stable configuration of D .

Let $C = \{V_0, V_1\}$ be a configuration for D such that u and v are not in the same group in C , and $S_i = \{s_i \in S \mid s_i \in V_i\}$, $i \in \{0, 1\}$. We can assume that S_0, S_1 is a partition of S . Let us call $T_i = \sum_{s_i \in S_i} s_i$. By definition of w , for any $i \in \{0, 1\}$, for all vertex s_i of S_i , we get that $f_{s_i}(C) = T + 1 - \sum_{s'_i \in S_i} s'_i + c_{s_i} = T - T_i + 1 + c_{s_i} = T_{1-i} + 1 + c_{s_i}$, where the value of $c_{s_i} \in \{0, s_i\}$ depends on whether z is in V_i . In the same way, one can prove that if z is in V_j for some $j \in \{0, 1\}$ then $f_z(C) = T_{1-j} + 1$.

It is important to notice that there is no subset of V that breaks the configuration C in creating a new group. Indeed, u and v cannot be part of such a subset, and it cannot contain z either, because z must be in the group of u or in the group of v . Furthermore, in any subset of S that is taken as a new group, all of the individual utilities are non positive, whereas they were positive in the original groups.

Let $s \in S$ be a vertex of V_i for some i , such that $T_i \leq T_{1-i}$. We get that $f_s(C) \geq T_{1-i} + 1$. Consequently, s has no interest in moving into V_{1-i} , whatever it is a collective move or not, because in such a new configuration C' we get that the new size of group V_i is $T'_i \leq T_i - s$, and $f_s(C') \leq T'_i + s + 1 \leq (T_i - s) + s + 1 = T_i + 1 \leq T_{1-i} + 1$. We prove in a similar way that vertex z has no interest in leaving the group V_i when $T_i \leq T_{1-i}$. Therefore if S_0, S_1 is a solution for the Partition problem, then $T_0 = T_1$, that is $T_i \leq T_{1-i}$ for all $i \in \{0, 1\}$, and so for every $k \geq 1$ we get that C is a k -stable configuration for D .

Conversely, let us suppose that S_0, S_1 is not a solution for the Partition problem, that is $T_i < T_{1-i}$ for some i , and let us show that C is not a 1-stable configuration for D . Obviously, z moves into V_i provided z belongs to V_{1-i} . So we now assume that z is in V_i . Then any singleton $\{s_{1-i}\}$ with $s_{1-i} \in S_{1-i}$ breaks the configuration, because in the new configuration C' where vertex s_{1-i} has moved into group V_i we get that $f_{s_{1-i}}(C') = T_{1-i} + 1$, that is strictly greater than $f_{s_{1-i}}(C) = T_i + 1$.

Finally, for any $k \geq 1$, there is a k -stable configuration C for the digraph D if and only if there is a solution to the Partition problem for the set S . \square

The same kind of pathological deviations arises with gossip. In this extension, which is still symmetrical, the nodes that are willing to merge their groups are allowed to do so. A first negative result has been proved in [9]: when $\mathcal{W} = \{-M, 1, N\}$ and N is sufficiently large, it is NP-hard to decide whether a given weighted graph admits a gossip-stable configuration. We prove the same result when $\mathcal{W} = \{-M, 0, 1\}$. Thus Theorem 8 shows that gossiping makes the stability problem intractable, even for slight extensions of the original model of [9]. For our reduction, we use 3-coloring problem, that consists in determining whether there exists a proper 3-coloring of a given graph $G = (V, E)$, a well known NP-complete problem [3].

THEOREM 8. *Let $\mathcal{W} = \{-M, 0, 1\}$. Deciding whether there exists a 2-stable, gossip-stable configuration for a given graph $G = (V, E, w)$ is NP-hard.*

PROOF. Let G be an instance for the 3-coloring problem. We define the graph $G_c = (V, E, w)$ as follows. For any

vertex v of G there are five vertices $v_1, v_2, v_{c_1}, v_{c_2}, v_{c_3}$ in V . We say that v_{c_i} is a colored vertex, and that it has the color i . There are also three special vertices in V , namely c_1, c_2 and c_3 .

For any vertex v of G we set $w(v_1, v_2) = 1$, $w(v_1, v_{c_1}) = w(v_1, v_{c_2}) = w(v_1, v_{c_3}) = 1$, $w(v_2, v_{c_1}) = w(v_2, v_{c_2}) = w(v_2, v_{c_3}) = 1$, and $w(v_{c_1}, v_{c_2}) = w(v_{c_1}, v_{c_3}) = w(v_{c_2}, v_{c_3}) = -M$. For any vertices u, v in G , and for any $i, j \in \{1, 2, 3\}$, $i \neq j$, we also set $w(c_i, v_{c_i}) = w(c_i, u_{c_i}) = 1$, $w(c_i, c_j) = -M$, and $w(u_{c_i}, c_j) = w(u_{c_i}, v_{c_j}) = -M$. In the case when u and v are adjacent in G , we set the weights $w(v_t, u_l) = -M$, for any $t, l \in \{1, 2\}$. Every other weight equals 0.

We have that, for any i , and for any vertex v of G , v_{c_i} and c_i gossip when they are not in the same group, because $w(v_{c_i}, c_i) = 1$ and their enemies are the same. Furthermore, vertices v_1 and v_2 gossip when they are not in the same group too. We now suppose there exists a gossip-stable configuration C such that none of the vertices v_{c_i} for $i \in \{1, 2, 3\}$ is in the same group as v_1 and v_2 , and we claim that there is a contradiction. Indeed, all the colored vertices that are in the same group as v_1 and v_2 , if any, have the same color i , because otherwise there would be enemies in the group and C would not be stable. Thus, there is at least one vertex v_{c_i} that breaks the configuration by joining the group of v_1 and v_2 , because $w(v_{c_i}, v_1) + w(v_{c_i}, v_2) = 2$ whereas $w(v_{c_i}, c_i) = 1$. A contradiction.

Finally, there can be only three groups in any gossip-stable configuration for G_c . Moreover, we claim those three groups are equivalent to a proper 3-coloring of G . Indeed, suppose C is a gossip-stable configuration for G_c , with only three groups. We have that neither c_1 and c_2 , nor c_2 and c_3 , nor c_3 and c_1 are in the same group in C , because they are enemies. Thus we can define a 3-coloring of G in the following way: for every vertex v of G , $c(v) = i$ if, and only if, v and c_i are in the same group in C , with $1 \leq i \leq 3$. This 3-coloring is proper, because otherwise there would be enemies in at least one group in C .

Conversely, suppose that we have a proper 3-coloring c of G . We define the configuration $C = \{V_1, V_2, V_3\}$ as follows. For every $1 \leq i \leq 3$, $c_i \in V_i$, and any colored vertex v_{c_i} is in V_i too. Furthermore, for all vertex v of G , we have that $v_1, v_2 \in V_{c(v)}$. By construction, there is no enemies in any of the three groups. Moreover, we claim that there is no gossiping between any two vertices that are in different groups. Indeed, there can only be gossiping between two vertices v_j and v_{c_i} , with $j \in \{1, 2\}$, $i \in \{1, 2, 3\}$, and $v_j \notin V_i$; however, the very definition of the configuration C implies that v_{c_i} has, at least, one enemy in the group of v_j , namely $c(v)$, and so, there is no gossiping at all. To conclude, we claim that any subset S , $|S| \leq 2$, does not break the configuration C . Indeed, the vertices in S can only found a new group in C , by the very definition of the weights w in G_c . Furthermore, for all vertex $u \in G_c$, we have that $f_u(C) \geq 1$. Hence, there is no vertex in G_c that can strictly increase its utility by a 2-deviation, and so, C is a gossip stable configuration. \square

4.2 Multi-modal relationship

Our model so far is heavily biased towards pairwise relationship, as the utility of a player depends on the sum of her interaction with all other members of the groups. In reality, more subtle interaction occur: one may be interested to interact with either friend u and v , but would not like to join a group where both of them are present.

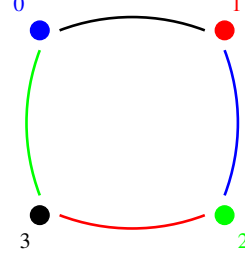


Figure 6: Counter-example for stability in the asymmetrical transitive model.

We show how the analysis we present generalize to this case. We define a coloring game over a weighted undirected hypergraph $H = (V, E, w)$, with w a weight function from $\mathcal{P}_t(V)$ to \mathbb{R} , where $\mathcal{P}_t(V)$ denotes the subsets in V of sizes at most t . Note that $t = \infty$ is allowed by our model. A configuration for H is a partition $C = (V_1, \dots, V_c)$ of V . The utility of a node u is now equal to $\sum_{\{u\} \subseteq S \subseteq C(u), |S| \leq t} w(S)$.

We prove in Lemma 13 that 1-stability is guaranteed. Interestingly, the proof uses a different potential function: $\phi(C) = \sum_{i=1}^c \sum_{S \subseteq V_i, |S| \leq t} w(S)$.

LEMMA 13. *For every hypergraph $H = (V, E, w)$, there exists a 1-stable configuration.*

Note that this assumed that players of a subset all receive the same gain. We also have proved that if that is not the case, the game is not stable even for $k = 1$.

We give the proof for those new degenerate cases when $t = 3$.

For every hyperedge $\{u_1, u_2, u_3\}$, let us denote by $w_{u_1}(u_2, u_3)$ the utility received by u_1 when u_2 and u_3 are in the same group than u_1 . That utility comes in addition to $w(u_1, u_2)$ and $w(u_1, u_3)$

LEMMA 14. *There exists a hypergraph $H = (V, E, w)$, with $\mathcal{W} = \{-M, 1\}$ and asymmetrical group relations, such that there does not exist any 1-stable configuration for H .*

PROOF. Consider the hypergraph $H = (V, E)$ depicted in Figure 6 with $V = \{u_0, u_1, u_2, u_3\}$ (the node labeled i in the figure corresponds to u_i). We set $w_{u_i}(u_i, u_j) = 1$ for any $i, j \in \{0, 1, 2, 3\}$, $i \neq j$. Furthermore, we assume that each node has a unique transitive restriction:

- $w_{u_i}(u_{i+1}, u_{i+2}) = -M$ ($i, i+1$, and $i+2$ are taken modulo 4);
- otherwise the transitive utility is 1.

The edges on Figure 6 represent the unique restriction for each of the four nodes. For example, the blue node u_0 does not want both u_1 and u_2 in its group because of the blue edge.

We will prove that there does not exist a 1-stable configuration C for this instance by considering all the cases.

First, if C is composed by a unique group, then the utility of each node is $-M$. Any node can forms its singleton group to get a utility of 1.

Consider any configuration C composed of one group of size 3 and one group of size 1. The first group is composed

of u_i , u_{i+1} , and u_{i+2} for some i (the values are taken modulo 4). Then, the utility of u_i is $-M$ because $w_{u_i}(u_{i+1}, u_{i+2}) = -M$. In that case, node u_i can strictly improve its utility by joining the group of u_{i-1} (its utility will be 2 instead of $-M$).

Consider any configuration C composed of two groups V_1 , V_2 of size 2. There are two cases. Consider first the case where $V_1 = \{u_i, u_{i+1}\}$ and $V_2 = \{u_{i+2}, u_{i+3}\}$ for some i (the value are taken modulo 4). In that case, utility of node u_i is 2 and can be 3 if u_i joins the other group V_2 . Indeed, recall that the unique transitive restriction for u_i is between u_{i+1} and u_{i+2} : $w_{u_i}(u_{i+1}, u_{i+2}) = -M$. The other case is when $V_1 = \{u_i, u_{i+2}\}$ and $V_2 = \{u_{i+1}, u_{i+3}\}$ for some i (the values are taken modulo 4). In that case, u_{i+2} can strictly improve its utility (from 2 to 3) by joining the other group V_2 . Again, the unique transitive restriction for u_{i+2} is between u_{i+3} and u_i .

Finally, if C is composed of four singleton groups, then any node can strictly improve its utility by joining any of the three other nodes (the utility is 2 instead of 1). \square

If subset of nodes have weights in $\mathcal{W} = \{-M, 1\}$ (nodes are either happy to interact to receive gain 1, or they wish to avoid at all cost to form a group containing this subset), all results on uniform games generalize. Formally, it holds because for any group $V_j \in C$, the vertices u in V_j have a utility $\sum_{\{u\} \subset S \subseteq V_j, |S| \leq t} w(S) = \sum_{i=1}^{t-1} \binom{|V_j|-1}{i}$, which is monotonically increasing in the size of their group.

If subsets of nodes have weights in $\mathcal{W} = \{-M, 0, 1\}$, there exists a k -stable configuration for a hypergraph H if the girth of the friendship hypergraph H^+ is at least $k+1$. Note that it does not imply 2-stability in general because the girth of a hypergraph may be 2.

Given a hypergraph $H = (V, E, w)$, we define the friendship hypergraph H^+ in the same way as the friendship graph in Section 3.1.

A Berge cycle in H^+ is a sequence $v_0, e_0, v_1, \dots, v_{p-1}, e_{p-1}, v_0$, such that:

- v_0, v_1, \dots, v_{p-1} are pairwise different vertices;
- e_0, e_1, \dots, e_{p-1} are pairwise different hyperedges;
- for all $0 \leq i \leq p-1$, $v_i, v_{i+1} \in e_i$.

LEMMA 15. *Let $H = (V, E)$ be an acyclic hypergraph, with c connected components. Then $|V| = \sum_{e \in E} (|e| - 1) + c$.*

PROOF. First we arbitrarily order the vertices in V , denoted v_0, v_1, \dots, v_{n-1} . We build an intersection graph $G = (V_H, E_H)$ as follows. $V_H = \{s_0, s_1, \dots, s_{n-1}\}$, such that for all $0 \leq i \leq n-1$, $s_i = \{e \in E : \{v_0, \dots, v_i\} \cap e = \{v_i\}\}$. Moreover, for all $0 \leq i < j \leq n-1$, the edge $s_i s_j = s_j s_i$ is in E_H if and only if there is $e \in s_i$ such that $v_j \in e$. Observe that such a hyperedge e is necessarily unique because H is acyclic. Let us call it $e_{i,j} = e_{j,i}$. Thus, we get $|E_H| = \sum_{e \in E} (|e| - 1)$. Moreover, we claim that G_H has the same number of connected components as H . Indeed, if s_i and s_j are adjacent in G_H then v_i and v_j both belong to $e_{i,j}$ in H by definition of G_H . Conversely, suppose that v_i and v_j are adjacent in H . Let e be a hyperedge such that $v_i, v_j \in e$. By definition of G_H , there exists $0 \leq i_0 \leq \min(i, j)$ such that $e \in s_{i_0}$. Furthermore, s_{i_0} is adjacent to both s_i and s_j . As a consequence, we get there are c connected components in G_H . Finally, we claim that G_H is acyclic, because any cycle

in G_H would give a Berge cycle in H . Therefore, we have $|V_H| = |E_H| + c$. \square

That easy result extends the well-known property of forests to the hypergraphs. We can now prove the following:

COROLLARY 3. *Given an integer $k \geq 1$ and a hypergraph $H = (V, E, w)$ with $\mathcal{W} = \{-M, 0, 1\}$, there exists a k -stable configuration for H if the girth of the friendship hypergraph H^+ is at least $k+1$.*

The proof is quite the same as for Theorem 5. Consequently, there always exists a 2-stable configuration if the friendship hypergraph is simple, that is the intersection of two different hyperedges has size at most 1.

4.3 Multichannels

Finally, one assumption of our model is that nodes form a partition, hence they are limited to a single channel to interact with their peers. In reality participants in social networks may engage in multiple groups.

Let $C = (V_1, V_2, \dots, V_c)$ be a configuration and let $C(u) = \{V_i, u \in C_i, 1 \leq i \leq c\}$ be the set of groups containing node u . The utility of u depends on the number of groups that u shares with each peer:

$$f_u(C) = \sum_{v \in V} h(|C(u) \cap C(v)|, w(u, v)), \quad (1)$$

where $h(g, w)$ is a function that measures the utility of sharing g groups with a node with weight w .

Note that we assume, without loss of generality, that

$$\begin{aligned} h(0, \cdot) &= 0, \quad h(\cdot, 0) = 0 \quad \text{and} \quad \forall w \in \mathbb{Z}, \quad h(1, w) = w, \\ \forall g \in \mathbb{N}, \quad w &\mapsto h(g, w) \text{ is a non-decreasing function,} \\ \forall w \in \mathbb{Z}, \quad g &\mapsto w \cdot h(g, w) \text{ is a non-decreasing function.} \end{aligned}$$

The last property simply ensures that $h(g, w)$ increases with g when w is positive, and decreases with g when w is negative. As an example: $h(g, w) = s(g) \cdot w$, for s a non-decreasing function that initially takes values 0 and 1, always satisfies these properties. Especially, when $s(g) = \mathbb{I}_{\{g > 0\}}$:

$$f_u(C) = \sum_{v \in V} \mathbb{I}_{\{C(u) \cap C(v) \neq \emptyset\}} \cdot w(u, v). \quad (2)$$

A configuration is said to use q channels if each user participate at most to q groups. We can assume that $|C(u)| = q$ for any u as a user can always form singleton groups.

Using the same potential function as before, *e.g.*, global utility, it follows that there always exists a 1-stable configuration with q channels. It is now natural to wonder what is the behavior of $k(\mathcal{W})$ when the number of channels is higher than 1. While it may have none or positive effects, we show that it is not always the case. Indeed, even for $\mathcal{W} = \{-M, 1\}$, Lemma 16 proves $k(\mathcal{W}) \leq 2$ with 2 channels while $k(\mathcal{W}) = \infty$ with single channel. The idea is to emulate a weight $w(u, v) = 0$ by forcing nodes u and v to be in a predetermined group.

LEMMA 16. *There exists a graph $G = (V, E, w)$ with $\mathcal{W} = \{-M, 1\}$ such that any configuration with 2 channels for G is not 3-stable.*

PROOF. We use the counter-example $G_{q=1} = (V_{q=1}, E_{q=1}, w_{q=1})$ of Lemma 10 for the proof. Note that, for each node $u \in V_{q=1}$, there exists at most one node $v \in V_{q=1}$ such that $w_{q=1}(u, v) = 0$.

Given a clique $K = (V, E)$ or order $p > 2|V_{q=1}|$, we set $w(u, v) = 1$ for any $u, v \in V$. We now assume that $\varepsilon < 1/p$ for the rest of the proof, and we construct the graph $G_{q=2} = (V_{q=2}, E_{q=2}, w_{q=2})$ as follows. For any node $u \in V_{q=1}$ such that $w_{q=1}(u, v) \neq 0$ for any $v \in V_{q=1}$, we add a copy of K , denoted $K_u = (V_u, E_u)$. We set $w_{q=2}(u, x) = 1$ for any $x \in V_u$. For any edge $(u, v) \in E_{q=1}$ such that $w_{q=1}(u, v) = 0$, we add another copy of K , denoted $K_{(u,v)} = (V_{(u,v)}, E_{(u,v)})$. We set $w_{q=2}(u, x) = w_2(v, x) = 1$ for any $x \in V_{(u,v)}$. Furthermore, we set $w_{q=2}(u, v) = 1$. Finally, for any graph K_u , with $u \in V_{q=1}$, we set $w_{q=2}(y, x) = -M$ for any $y \in V_{q=1} \setminus \{u\}$ and for any $x \in V_u$. Similarly, for any graph $K_{(u,v)}$, with $(u, v) \in E_{q=1}$, we set $w_{q=2}(y, x) = -M$ for any $y \in V_{q=1} \setminus \{u, v\}$ and for any $x \in V_{(u,v)}$.

First, we get that for any $u, v \in V_{q=1}$ such that $w(u, v) = 0$, for any $x, y \in V_{(u,v)}$, $|C(x) \cap C(y)| = 2$ in any 3-stable configuration $C = (V_1, \dots, V_c)$ with 2 channels for $G_{q=2}$.

As $\varepsilon M > n$, we have that any $x \in V_{(u,v)}$ cannot be in a group containing any node $y \in V_{q=2} \setminus (E_{(u,v)} \cup \{u, v\})$.

We first prove that in any 3-stable configuration $C = (V_1, \dots, V_c)$ with 2 channels for $G_{q=2}$, we have $|C(x) \cap C(y)| \neq 1$, for any $x, y \in V_{(u,v)}$. By contradiction. Assume there exist $x, y \in V_{(u,v)}$ such that $|C(x) \cap C(y)| = 1$. Without loss of generality, assume that $f_x(C) \geq f_y(C)$. Then, there exists a 1-deviation: y reaches the group V_i such that $x \in V_i$ and $y \notin V_i$. In that case, $f_y(C') > f_y(C)$, where C' is the configuration for $G_{q=2}$ after the previous 1-deviation. Indeed, $f_y(C') = f_x(C) + \varepsilon$.

Thus, we assume that, in any 3-stable configuration $C = (V_1, \dots, V_c)$ with 2 channels for $G_{q=2}$, we have $|C(x) \cap C(y)| \neq 1$, for any $x, y \in V_{(u,v)}$. Furthermore, if there exist $x, y \in V_{(u,v)}$ such that $|C(x) \cap C(y)| = 0$, then there exist two groups V_{x_1} and V_{x_2} such that $x \in V_{x_1}$, $x \in V_{x_2}$, and $V_{x_1} = V_{x_2}$, and there exist two groups V_{y_1} and V_{y_2} such that $y \in V_{y_1}$, $y \in V_{y_2}$, and $V_{y_1} = V_{y_2}$. Thus, $f_x(C) = (1 + \varepsilon)(|V_{x_1}| - 1)$, and $f_y(C) = (1 + \varepsilon)(|V_{y_1}| - 1)$. Without loss of generality, suppose $f_x(C) \geq f_y(C)$. Then, there exists a 1-deviation: y leaves V_{y_1} and reaches V_{x_1} . We get $f_y(C') = |V_{x_1}| + |V_{y_2}| - 1 > f_y(C)$ because $\varepsilon < \frac{1}{p}$. Thus, in any 3-stable configuration $C = (V_1, \dots, V_c)$ with 2 channels for $G_{q=2}$, we have for any $x, y \in V_{(u,v)}$, $|C(x) \cap C(y)| = 2$, and so, the claim is proved.

Then, by the choice of $0 < \varepsilon < 1/p$ with $p > 2|V_1|$, there exist two groups V_i and V_j , $1 \leq i < j \leq c$, such that $V_i = V_{(u,v)}$ and $V_j = V_{(u,v)} \cup \{u, v\}$. Indeed, otherwise there exists a 1-deviation for C , because by construction $|V_{(u,v)}| > f_u(C')$, with C' being any configuration with 2 channels for $G_{q=1}$. We obtain the same result for copies K_u . To summarize, after setting the groups from the previously necessary conditions for having a 3-stable configuration with 2 channels for $G_{q=1}$, each node $u \in V \setminus V_{q=1}$ belongs to exactly 2 groups, and each node $u \in V_{q=1}$ belongs to exactly 1 group.

Thus, there exists a 3-stable configuration for $G_{q=2}$ with 2 channels if, and only if, there exists a 3-stable configuration for $G_{q=1}$ with single channel. Indeed, any two nodes $u, v \in V_{q=1}$ such that $w_{q=1}(u, v) = 0$, are already in a same group. Thus, by the choice of the function h , this mimics a weight ε between these two nodes for determining the other groups, that is quite the same as a weight 0, because ε is arbitrarily small (see Corollary 2). Finally, there does not exist a 3-stable configuration for $G_{q=2}$ with 2 channels because there

does not exist a 3-stable configuration for $G_{q=1}$ with single channel by Lemma 10. \square

In the case of general weights, a stability criterion depending on q has yet to be found. We prove in Theorem 9 that any chaotic behavior may be observed between any two consecutive numbers of channels.

LEMMA 17. *Let $W = \{-M, -4, 2, 6, 7\}$. There exists a graph $G = (V, E, w)$ such that any configuration C with single channel for G is not 2-stable.*

PROOF. Let $G_1 = (V, E, w)$ be the graph built as follows. Let $V = \{u_1, u_2, u_3, u_4\}$. We set $w(u_1, u_2) = w(u_1, u_3) = -M$, $w(u_1, u_4) = 7$, $w(u_2, u_4) = 6$, $w(u_3, u_4) = 2$, and $w(u_2, u_3) = -4$.

For any $k \geq 1$, any k -stable configuration C with single channel for G_1 is such that $f_C(u) \geq 0$ for any $u \in V$. Thus, for any $k \geq 1$, the following configurations for G_1 are not k -stable:

- $C = (\{u_1\}, \{u_2, u_3, u_4\})$ as $f_{u_3}(C) = -2 < 0$;
- $C = (\{u_1\}, \{u_2, u_3\}, \{u_4\})$ as $f_{u_3}(C) = -4 < 0$;
- $C = (\{u_1, u_4\}, \{u_2, u_3\})$ as $f_{u_3}(C) = -4 < 0$;
- any configuration $C = (V_1, V_2, V_3, V_4)$ (V_i may be empty for some i , $1 \leq i \leq 4$) such that $u_1, u_2 \in V_i$ and/or $u_1, u_3 \in V_i$ for some i , $1 \leq i \leq 4$.

Thus, it remains to prove the result for the other configurations for G_1 .

- Consider the configuration $C = (\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\})$ for G_1 . C is not 1-stable because u_4 can reach group $\{u_1\}$. Indeed, $f_{u_4}(C') = f_{u_4}(C) + 7 = 7$ where $C' = (\{u_1, u_4\}, \{u_2\}, \{u_3\})$ is the resulting configuration with single channel for G_1 after this 1-deviation.
- Consider the configuration $C = (\{u_1, u_4\}, \{u_2\}, \{u_3\})$ for G_1 . C is not 2-stable because u_2 and u_4 can reach group $\{u_3\}$. Indeed, $f_{u_2}(C') = f_{u_2}(C) + 6 - 4 = f_{u_2}(C) + 2 = 2$ and $f_{u_4}(C') = f_{u_4}(C) - 7 + 6 + 2 = f_{u_4}(C) + 1 = 8$ where $C' = (\{u_1\}, \{u_2, u_3, u_4\})$ is the resulting configuration for G_1 after this 2-deviation.
- Consider the configuration $C = (\{u_1\}, \{u_2, u_4\}, \{u_3\})$ for G_1 . C is not 1-stable because u_4 can reach group $\{u_1\}$. Indeed, $f_{u_4}(C') = f_{u_4}(C) - 6 + 7 = f_{u_4}(C) + 1 = 7$ where $C' = (\{u_1, u_4\}, \{u_2\}, \{u_3\})$ is the resulting configuration for G_1 after this 1-deviation.
- Consider the configuration $C = (\{u_1\}, \{u_2\}, \{u_3, u_4\})$ for G_1 . C is not 1-stable because u_4 can reach group $\{u_1\}$. Indeed, $f_{u_4}(C') = f_{u_4}(C) - 2 + 7 = f_{u_4}(C) + 5 = 7$ where $C' = (\{u_1, u_4\}, \{u_2\}, \{u_3\})$ is the resulting configuration for G_1 after this 1-deviation.

Finally, any configuration C with single channel for G_1 is not a 2-stable configuration for G_1 . \square

COROLLARY 4. *Let $W = \{-M, -4, 2, 6, 7\}$. Given an integer $q \geq 1$, there exists a graph $G_q = (V_q, E_q, w_q)$ such that any configuration C with q channels for G_q is not 2-stable, whereas there always exists a 2-stable configuration C' with $q' \neq q$ channels for G_q .*

PROOF. We arbitrarily select $\epsilon < 1/N$. Let $G_1 = (V_1, E_1, w_1)$ be the graph that is presented in the proof of Lemma 17 (we re-use the same notations for the vertices in G_1). We build the graph G_q as follows. First, the set of vertices is $V_q = V_1 \cup \{x_1, \dots, x_{q-1}\}$. For any $u, v \in V_1$, $w_q(u, v) = w_1(u, v)$. Furthermore, for any $1 \leq i < j \leq q-1$, $w(x_i, u_4) = w(x_j, u_4) = N$, $w(x_i, x_j) = w(x_i, v) = w(x_j, v) = -M$ for any $v \in V_1 \setminus \{u_4\}$.

We have that, for any $q' < q$, $C_{q'} = (\{u_4, x_1\}, \dots, \{u_4, x_{q'}\}, \{u_1\}^1, \dots, \{u_1\}^{q'}, \{u_2\}^1, \dots, \{u_2\}^{q'}, \{u_3\}^1, \dots, \{u_3\}^{q'}, \{x_1\}^1, \dots, \{x_1\}^{q'-1}, \dots, \{x_{q'}\}^1, \dots, \{x_{q'}\}^{q'-1}, \{x_{q'+1}\}^1, \dots, \{x_{q'+1}\}^{q'}, \dots, \{x_{q-1}\}^1, \dots, \{x_{q-1}\}^{q'})$ is a 2-stable configuration with q' channels for G_q .

However, in any stable configuration C_q with q channels for G_q , we necessarily have $\{u_4, x_i\}$ as a group in C_q (without repetitions), for any $1 \leq i \leq q-1$. As there does not exist any 2-stable configuration with single channel for G_1 , there does not exist any 2-stable configuration with q channels for G_q .

If $q' = q+1$, then $C_{q'} = (\{u_4, x_1\}, \dots, \{u_4, x_{q-1}\}, \{u_4, u_1\}, \{u_4, u_2\}, \{u_1\}^1, \dots, \{u_1\}^{q'-1}, \{u_2\}^1, \dots, \{u_2\}^{q'-1}, \{u_3\}^1, \dots, \{u_3\}^{q'}, \{x_1\}^1, \dots, \{x_1\}^{q'-1}, \dots, \{x_{q-1}\}^1, \dots, \{x_{q-1}\}^{q'-1})$ is a 2-stable configuration with q' channels for G_q .

Finally, if $q' \geq q+2$, then $C_{q'} = (\{u_4, x_1\}^1, \dots, \{u_4, x_1\}^{q'-q-1}, \{u_4, x_2\}, \dots, \{u_4, x_{q-1}\}, \{u_4, u_1\}, \{u_4, u_2\}, \{u_4, u_3\}, \{u_1\}^1, \dots, \{u_1\}^{q'-1}, \{u_2\}^1, \dots, \{u_2\}^{q'-1}, \{u_3\}^1, \dots, \{u_3\}^{q'-1}, \{x_1\}^1, \dots, \{x_1\}^{q'+1}, \{x_2\}^1, \dots, \{x_2\}^{q'-1}, \dots, \{x_{q-1}\}^1, \dots, \{x_{q-1}\}^{q'-1})$ is a 2-stable configuration with q' channels for G_q . \square

THEOREM 9. *For any sequence of positive integers q_1, q_2, \dots, q_p , and for any sequence $1 \leq i_1 < i_2 < \dots < i_l \leq p$, there exists a graph $G = (V, E, w)$ such that there is a 2-stable configuration with q_j channels for G if, and only if, $j \neq i_t$, for every $1 \leq t \leq l$.*

PROOF. For any $1 \leq t \leq l$, let $G_{q_{i_t}}$ be the graph in the proof of Corollary 4 such that there exists a 2-stable configuration with q' channels for $G_{q_{i_t}}$ if, and only if, $q' \neq q_{i_t}$. By convention, G_0 is any graph such that there always is a 2-stable configuration with q channels for G_0 ; for instance, we may assume that all the weights for G_0 are positive.

Then we build a graph G from $G_0, G_{q_1}, \dots, G_{q_{i_l}}$, such that any vertices u, v that are not in the same graph G_q are enemies (e.g. $w(u, v) = -M$). In so doing, every graph G_q is independent of the other ones, and so, there is a 2-stable configuration with q' channels for G if, and only if, there exists a 2-stable configuration with q' channels for every $G_{q_{i_t}}$, $1 \leq t \leq l$. \square

5. EFFICIENCY STABILITY TRADE-OFF

Efficiency of configuration can be estimated through the sum of utility received by all the players. It has been proved in [9], that computing a maximum configuration is NP-complete even with $\mathcal{W} = \{-M, 1\}$. Hence, it does not only require coordination, but also has a prohibitive computational cost. We first strengthen this result by showing that this optimum cannot even be approximated within a polynomial multiplicative factor.

LEMMA 18. *For every $1 > \epsilon > 0$, the problem of finding a maximum configuration cannot be approximated within any $n^{\epsilon-1}$ -ratio in polynomial time, unless $P=NP$.*

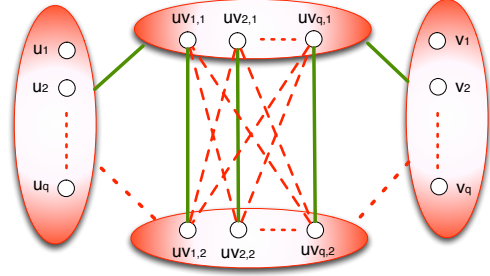


Figure 7: Representation in the graph G' of an edge (u, v) with positive weight in Definition 9.

PROOF. Let us constraint ourselves to the set of weights $\{-M, 1\}$. By contradiction, suppose there exists a polynomial-time algorithm A , such that for any graph G we have $f(A(G)) \geq n^{\epsilon-1}OPT(G)$. We can use algorithm A to compute an independent set of the conflict graph of G . To do this, it suffices to take the largest group in $A(G)$. Moreover, we know by [8], the MAXIMUM INDEPENDENT SET PROBLEM cannot be approximated within any $n^{\frac{5}{2}-1}$ -ratio in polynomial time, unless $P=NP$. Then, if $P \neq NP$, there are graphs G such that the largest group in $A(G)$ has size lesser than $n^{\frac{5}{2}-1}|MIS(G)|$, where $MIS(G)$ denotes a maximum independent set of the conflict graph of G . Especially, for such a graph G , we have $f(A(G)) < n \cdot (n^{\frac{5}{2}-1}|MIS(G)| - 1)n^{\frac{5}{2}-1}|MIS(G)| < n^{\epsilon-1}(|MIS(G)| - 1)|MIS(G)|$. However, the utility of the configuration with single group $\{MIS(G)\}$ for G is already $(|MIS(G)| - 1)|MIS(G)|$. So, we have $f(A(G)) < n^{\epsilon-1}OPT(G)$, which is a contradiction unless $P = NP$. \square

However, we show that using $q > 1$ channels adds in general no particular difficulty to that computation. It could be useful for classes of graphs for which we have a good approximation to find a maximum configuration.

DEFINITION 9. *For any graph $G = (V, E, w)$ with $\mathcal{W} \subseteq \{-M\} \cup \mathbb{N}$, let $G' = (V', E', w')$ be as follows.*

- a) *For any $u \in V$, we have q nodes $u_1, u_2, \dots, u_q \in V'$ and we set $w'(u_i, u_j) = -M$ for any $i, j, 1 \leq i < j \leq q$;*
- b) *For any $u, v \in V$ such that $w(u, v) \in \{-M, 0\}$, we set $w'(u_i, v_j) = w(u, v)$ for any $i, j, 1 \leq i, j \leq q$;*
- c) *For any $u, v \in V$ such that $w(u, v) > 0$, for any $g, 1 \leq g \leq q$, we have two nodes $uv_{g,1}, uv_{g,2} \in V'$, as shown in Figure 7, such that for any $i, j, 1 \leq i, j \leq q$:*

- $w'(u_i, uv_{g,1}) = w'(v_i, uv_{g,1}) = (h(g, w(u, v)) - h(g-1, w(u, v)))/2$;
- $w'(u_i, v_j) = 0$;
- $w'(uv_{g,1}, uv_{g,2}) = 3(h(g, w(u, v)) - h(g-1, w(u, v)))/4$;
- $w'(y, uv_{g,2}) = -M, \forall y \notin \{uv_{g,1}, uv_{g,2}\}$;
- $w'(uv_{g,1}, uv_{g',1}) = -M, \forall g' \neq g, 1 \leq g' \leq q$;
- $w'(y, uv_{g,1}) = 0, \forall y \notin \{u_i, v_i, uv_{g',1}, uv_{g,2}, 1 \leq i, g' \leq q\}$.

Let C' be a maximum single channel configuration in G' . Note that $|C'(u_i) \cap C'(u_j)| = 0$ for any $i \neq j$, and $|C'(u_i) \cap C'(v_j)| = 1$ if, and only if, a unique $uv_{g,1}$ exists such that $|C'(u_i) \cap C'(uv_{g,1}) \cap C'(v_j)| = 1$. From C' we can then build a configuration C with q channels by replacing all nodes u_i by u and deleting all nodes $uv_{x,1}$ and $uv_{x,2}$. The proof concludes by proving that C' is not maximum if C isn't. Formally:

THEOREM 10. *Assume h concave. From a maximum configuration with single channel for G' , we deduce in linear time a maximum configuration with q channels for G .*

PROOF. Let $C = (V_1, \dots, V_c)$ be any maximum configuration with single channel for G' . Note that for any $u, v \in V'$, then $|C(u_i) \cap C(u_j)| \in \{0, 1\}$. We prove several properties for C .

a) For any $u \in V$, we have $|C(u_i) \cap C(u_j)| = 0$ for any $i, j, 1 \leq i < j \leq q$, because $w'(u_i, u_j) = -M$.

b) For any $u, v \in V$ such that $w(u, v) = -M$, we have $|C(u_i) \cap C(v_j)| = 0$ for any $i, j, 1 \leq i, j \leq q$, because $w'(u_i, v_j) = -M$.

c) For any $u, v \in V$ such that $w(u, v) > 0$, if there exist $i, j, 1 \leq i, j \leq q$, such that $|C(u_i) \cap C(v_j)| = 1$, then there exists $x, 1 \leq x \leq q$, such that $|C(u_i) \cap C(v_j) \cap C(uv_{x,1})| = 1$. By contradiction. If $|C(u_i) \cap C(v_j)| = 1$ and $|C(u_i) \cap C(v_j) \cap C(uv_{x,1})| = 0$ for any $x, 1 \leq x \leq q$, then there exists a configuration C' with single channel for G' such that $f(C') > f(C)$. Indeed, we proved before that $|C(u_{i'}) \cap C(u_{j'})| = 0$ and $|C(v_{i'}) \cap C(v_{j'})| = 0$ for any $i', j', 1 \leq i' < j' \leq q$. Thus, there exists $x, 1 \leq x \leq q$, such that $|C(uv_{x,1}) \cap C(u_{i'})| = 0$ and $|C(uv_{x,1}) \cap C(v_{i'})| = 0$ for any $i', 1 \leq i' \leq q$. We get two cases:

1. if $|C(uv_{x,1}) \cap C(y)| = 0$ for any $y \in V'$, then we add $uv_{x,1}$ in the group of u_i and v_j and we get $f(C') = f(C) + 2(h(x, w(u, v)) - h(x-1, w(u, v)))$.
2. if $|C(uv_{x,1}) \cap C(uv_{x,2})| = 1$, then $|C(uv_{x,1}) \cap C(y)| = 0$ for any $y \in V'$ by construction of G' . Thus, we add $uv_{x,1}$ in the group of u_i and v_j and we get $f(C') = f(C) + 2(h(x, w(u, v)) - h(x-1, w(u, v))) - 6(h(x, w(u, v)) - h(x-1, w(u, v)))/4 > f(C)$.

d) For any $u, v \in V$ such that $w(u, v) > 0$, if there exist $i, j, x, 1 \leq i, j, x \leq q$, such that $|C(u_i) \cap C(v_j) \cap C(uv_{x,1})| = 1$, then for any $x' < x$, there exists $i', j', 1 \leq i', j' \leq q$, such that $|C(u_{i'}) \cap C(v_{j'}) \cap C(uv_{x',1})| = 1$. By contradiction. If there exists $x' < x$ such that $|C(u_{i'}) \cap C(v_{j'}) \cap C(uv_{x',1})| = 0$ for any $i', j', 1 \leq i', j' \leq q$, then there exists a configuration C' with single channel for G' such that $f(C') \geq f(C)$. As h is concave, we have $h(x', w(u, v)) - h(x'-1, w(u, v)) \geq h(x, w(u, v)) - h(x-1, w(u, v))$ because $x' < x$. There are three cases.

1. Without loss of generality, if for some $i', 1 \leq i' \leq q$, $|C(u_{i'}) \cap C(uv_{x',1})| = 1$, and so $|C(v_{j'}) \cap C(uv_{x',1})| = 0$ for any $j', 1 \leq j' \leq q$, then we add $uv_{x,1}$ to the group of u_i and v_j , and we add $uv_{x,1}$ to the group with $uv_{x,2}$. We get the configuration C' such that $f(C') = f(C) - (h(x', w(u, v)) - h(x'-1, w(u, v))) + 2(h(x', w(u, v)) - h(x'-1, w(u, v))) - 2(h(x, w(u, v)) - h(x-1, w(u, v))) + 6(h(x, w(u, v)) - h(x-1, w(u, v)))/4 > f(C)$ because $|C(uv_{x',1}) \cap C(uv_{x',2})| = 0$ and because h is such that $h(x', w(u, v)) - h(x'-1, w(u, v)) \geq h(x, w(u, v)) - h(x-1, w(u, v))$.
2. If $|C(u_{i'}) \cap C(uv_{x',1})| = 0$, $|C(v_{j'}) \cap C(uv_{x',1})| = 0$ for any $i', j', 1 \leq i', j' \leq q$, and $|C(uv_{x',1}) \cap C(uv_{x',2})| = 0$, then we add $uv_{x,1}$ to the group of u_i and v_j , and we add $uv_{x,1}$ to the group with $uv_{x,2}$. We obtain the configuration C' such that $f(C') = f(C) - 2(h(x, w(u, v)) - h(x-1, w(u, v))) + 2(h(x', w(u, v)) - h(x'-1, w(u, v))) + 6(h(x, w(u, v)) - h(x-1, w(u, v)))/4 > f(C)$ by the choice of h .
3. If $|C(u_{i'}) \cap C(uv_{x',1})| = 0$, $|C(v_{j'}) \cap C(uv_{x',1})| = 0$ for any $i', j', 1 \leq i', j' \leq q$, and $|C(uv_{x',1}) \cap C(uv_{x',2})| = 1$,

then we add $uv_{x',1}$ to the group of u_i and v_j , and we add $uv_{x,1}$ to the group with $uv_{x,2}$. We obtain the configuration C' such that $f(C') = f(C) - 2(h(x, w(u, v)) - h(x-1, w(u, v))) + 2(h(x', w(u, v)) - h(x'-1, w(u, v))) - 6(h(x', w(u, v)) - h(x'-1, w(u, v)))/4 + 6(h(x, w(u, v)) - h(x-1, w(u, v)))/4 > f(C)$ by the choice of h .

To summarize, any maximum configuration C for G' is such that: for any $u \in V$, $|C(u_i) \cap C(u_j)| = 0$ for any $i, j, 1 \leq i, j \leq q$; for any $u, v \in V$ such that $w(u, v) = -M$, $|C(u_i) \cap C(v_j)| = 0$ for any $i, j, 1 \leq i, j \leq q$; for any $u, v \in V$ such that $w(u, v) > 0$, then there exists $x, 0 \leq x \leq q$, such that for any $x' \leq x$, there exists $i, j, 1 \leq i, j \leq q$, such that $|C(u_i) \cap C(v_j) \cap C(uv_{x',1})| = 1$, and for $x' > x$, $|C(uv_{x',1}) \cap C(uv_{x',2})| = 1$ and so $|C(uv_{x',1})| = 2$.

We get that any maximum configuration C for G' is such that $f(C) = c(C) + f'(C)$ where $c(C)$ is the constant $\sum_{u,v \in V, w(u,v) > 0} \sum_{x=1}^q 6(h(x, w(u, v)) - h(x-1, w(u, v)))/4$ and $f'(C) = \sum_{u,v \in V, w(u,v) > 0} \sum_{x=1}^q \mathbb{I}_{\{\exists i,j, |C(u_i) \cap C(v_j)|=1\}} 2(h(x, w(u, v)) - h(x-1, w(u, v)))/4$.

We now construct the corresponding configuration C_q for G . For any $V_j \in C$, we construct the group $V_{j'} \in C_q$ as follows. For any $u \in V$, $u \in V_{j'}$ if, and only if, there exists $i, 1 \leq i \leq q$, such that $u_i \in V_j$. Clearly, the construction of G' and the construction of C_q can be done in linear time on the size of G .

By construction, we get that $f(C_q) = 4f'(C)$.

We now prove that if C with single channel is maximum for G' , then C_q with q channels is maximum for G . Suppose C is maximum for G' . By contradiction. Suppose there exists a configuration C'_q with q channels for G such that $f(C'_q) > f(C_q)$. We prove that there exists a configuration C' with single channel for G' such that $f(C') > f(C)$. We construct C' as follows. For any $V_{i_q} \in C'_q$, we construct the group $V_{i'_q}$ for C' as follows: sequentially, for any $u, v \in V_{i_q}$, there exist $i, j, 1 \leq i, j \leq q$ such that u_i and u_j are not yet in one group in C' . If there exist, we choose u_i and/or v_j that are yet in group $V_{i'_q}$. Furthermore, if $w(u, v) > 0$, we choose the smallest $x, 1 \leq x \leq q$, such that $uv_{x,1}$ is not yet in one group in C' . We add u_i, u_j and $uv_{x,1}$ in group $V_{i'_q}$.

We get that $f(C') = c(C') + f'(C')$ where $c(C') = c(C)$ and $f'(C') = f'(C'_q)/4$ by construction.

Thus, we get $f(C') > f(C)$ because $f'(C') > f'(C)$. A contradiction because C is maximum for G' . Finally, C_q with q channels is maximum for G . \square

Note that, in our extended model with multichannels, the total utility of the players can only improve as the number of channels q gets larger, because more configurations are available. Unfortunately, we also prove that computing the minimum number of channels that guarantee a given threshold global utility is hard to approximate.

THEOREM 11. *Given $U \geq 1$, computing q^* , the minimum q for which there exists a configuration C satisfying $f(C) \geq U$, cannot be approximated within $|V|^{1-\varepsilon}$ for any $\varepsilon > 0$ in polynomial time, unless $P=NP$.*

PROOF. We construct the graph $G = (V, E, w)$ as follows. Let $G_1 = (V_1, E_1, w_1)$ be any graph such that $w_1(u, v) \in \{-M, 0\}$ for any $u, v \in V_1$. Let $V = V_1 \cup \{u, v_1, v_2, \dots, v_{|V_1|}\}$. We set $w(u, x) = N$ for any $x \in V_1$. We set $w(u, v_i) = 1$ for any $i, 1 \leq i \leq |V_1|$. We set $w(v_i, x) = -M$ for any $x \in V_1$ and for any $i, 1 \leq i \leq |V_1|$.

Let $U = 2|V_i|$. Thus, if there exists a configuration C for G with q channels such that $f(C) \geq U$, then $|C(u) \cap C(x)| \geq 1$ for any $x \in V \setminus \{u\}$. Indeed there are exactly $2|V_i|$ edges with weights 1. Thus, by construction of G , the problem remains to minimize the number of groups containing node u . Clearly by construction, we have $|V_i|$ groups of size 2 composed of $\{u, v_i\}$ for any i , $1 \leq i \leq |V_i|$. Then, u must be in one group with each $x \in V_1$. As $w_1(u, v) \in \{-M, 0\}$ for any $u, v \in V_1$, then we have to partition the nodes V_1 into groups without arcs with weight $-M$, and we have to minimize the number of groups. Thus, the problem remains to find the chromatic number of the graph induced by arcs with weight $-M$. As for all $\varepsilon > 0$, approximating the chromatic number within $n^{1-\varepsilon}$ is NP-hard [12], then we get the result. \square

However, in spite of this complexity, we prove that collisions and multiple channels have in general a beneficial effect on the price of anarchy, which is the ratio of the worst stable equilibrium to this optimal. Nash Equilibria obtained by the dynamic of the system satisfy a general bound that improves with q , in contrast with worst Nash Equilibria that are always arbitrarily bad. Finally, we show that higher order stability, although it is significantly harder to obtain in general, potentially improve this bound significantly.

5.1 Efficiency of Nash Equilibrium

Since we proved, there is few hope we can compute efficient configurations in polynomial time, our goal is now to study the efficiency of any stable configuration, and especially the ones that we can compute by using our dynamic system.

We start by studying the efficiency of the largest class of stable configurations: Nash Equilibria ($k = 1$). For any graph G , we denote by C^+ a maximum configuration; the *worst case* C_k^- denotes a configuration with the minimum utility that is obtained in a k -stable configuration using q channels. We first show in Lemma 19 that Nash Equilibria can be arbitrarily bad:

LEMMA 19. *Let $a, b > 0$ be two integers. With $\mathcal{W} \supseteq \{-M, 0, b\}$ or $\mathcal{W} \supseteq \{-a, b\}$, for any $R \geq 1$, there are graphs $G = (V, E, w)$ with $f(C^+) \geq R$ and $f(C_1^-) = 0$.*

PROOF. a) Case $\{-M, 0, b\} \subseteq \mathcal{W}$.

Let $n' \geq \sqrt{R/2b}$ be an integer. Let $G = (V, E, w)$ with $V = V_1 \cup V_2 \cup \{v_1, v_2\}$, such that $|V_1| = |V_2| = n'$. For any $u, v \notin \{v_1, v_2\}$, we set $w(u, v) = b$ if u and v are not in the same subset V_i , and $w(u, v) = 0$ otherwise. Moreover, v_1 (v_2 , respectively) is enemy with any vertex of V_2 (V_1 , respectively), and for any $u \in V_1$, any $v \in V_2$, we have $w(u, v_1) = w(v, v_2) = w(v_1, v_2) = 0$. Configuration C containing $(V_1 \cup \{v_1\}, V_2 \cup \{v_2\})$ q times is 1-stable for G and $f(C) = 0$.⁴ Furthermore, a maximum configuration with single channel for G is $C^+ = (V_1 \cup V_2, \{v_1, v_2\})$ and $f(C^+) = 2bn'^2 \geq R$.

b) Case $\{-a, b\} \subseteq \mathcal{W}$.

Let $n' \geq \frac{R}{2b(b+a)}$ be an integer. Let $d = 2n'b$, and $n = 2n'(b+a) + 1$. First observe that $n > d$. We claim that there exists a d -regular graph G with n vertices. Indeed, let us choose G such that the vertices in G are labeled by the

⁴This holds, for instance, if we choose $h(0, w) = 0$, $h(g + 1, w) = w$ for the proof.

integers of \mathbb{Z}_n ; for all $i, j \in \mathbb{Z}_n$, there is an edge (i, j) in G if, and only if, $j \equiv i - d/2 + \alpha \pmod{n}$, for every $0 \leq \alpha \leq d$ such that $\alpha \neq d/2$.

We now define the graph $G' = (V, E, w)$ as follows. The vertices in G' are the vertices in G . For all $i, j \in \mathbb{Z}_n$, $i \neq j$, $w(i, j) = -a$ if (i, j) is an edge of G , and $w(i, j) = b$ otherwise.

We have that $C = (V)$ is a 1-stable configuration with single channel for G' . Furthermore, for all $i \in \mathbb{Z}_n$, $f_i(C) = -da + (n - d - 1)b = 0$.

On the other hand, $C_p = (\{0, 1\}, \{2, 3\}, \dots, \{n-3, n-2\}, \{n-1\})$ is a configuration with single channel for G' such that $f(C_p) = b(n-1) \geq R$. \square

On the other hand, such situation is an oddity as precised in Lemma 20. We denote by C_1^{d-} the worst equilibrium obtained under the dynamic of the system. Recall that w_p is the maximum weight of \mathcal{W} .

LEMMA 20. *Given integers $q, n \geq 1$ and a set \mathcal{W} such that $\mathcal{W}^+ \neq \emptyset$, for any graph $G = (V, E, w)$ such that $|V| \leq n$, we have $f(C^+)/f(C_1^{d-}) = O(h(q, w_p)n)$.*

PROOF. Let $G = (V, E, w)$ be a graph with m_+ the number of edges with positive weights. We choose G as an input for the greedy algorithm. It takes $s \geq 0$ steps for computing a 1-stable configuration C_s with q channels for G . Note that $f(C_s) \geq s$ because the global utility strictly increases at each step (Theorem 4).

Let us denote V_s the set of nodes u such that there is at least one of their q groups in C_s that is not a singleton group. We prove that $n_s \leq 2s$, where $n_s = |V_s|$. Indeed, at each step of the algorithm, any vertex v has at most one of its singleton groups that has been modified into another, larger group. Furthermore, any such step modifies the singleton groups of at most two distinct vertices.

As a consequence, there are at most $nn_s \leq 2sn$ edges (u, v) in A such that $w(u, v) > 0$ and either $u \in V_s$ or $v \in V_s$. Moreover, note that there does not exist any couple of vertices $u, v \in V \setminus V_s$, such that $w(u, v) > 0$, because C_s is 1-stable. Hence, $2sn \geq m_+$, and so $f(C^+)/f(C_s) \leq 2h(q, w_p)m_+/s = O(h(q, w_p)n)$. \square

The upper-bound above, if it were tight, would imply that increasing of the number of channels might deteriorate the quality of the lowest computable Nash Equilibria. On the contrary, we prove in Lemma 21 that the number of channels may have a positive impact for some sets of weights.

LEMMA 21. *Given integers $q, n \geq 1$ and a set $\mathcal{W} = \mathcal{W}^+ \cup \{-M\}$ such that $\mathcal{W}^+ \neq \emptyset$, for any graph G with at most n nodes, we have $f(C^+)/f(C_1^{-,g'}) = O((1 + 2n/q)h(q, w_p))$.*

PROOF. Notations are the same as for the previous proof of Lemma 20. We set $m_+ = m_{1,+} + m_{0,+}$, where $m_{1,+}$ is the number of edges (u, v) such that $w(u, v) > 0$ and $C_s(u) \cap C_s(v) \neq \emptyset$, and $m_{0,+}$ is the number of edges (u', v') such that $w(u', v') > 0$ whereas $C_s(u') \cap C_s(v') = \emptyset$, respectively. By induction on the number of steps of the greedy algorithm, we get that for every two vertices u, v such that $w(u, v) = -M$, $C_s(u) \cap C_s(v) = \emptyset$, because the constant M is large enough so that u would avoid at any cost to interact with v , and reciprocally. As a consequence, we have $f(C_s) \geq \max(s, m_{1,+})$. Furthermore, we denote V_q the set of vertices such that none of their q groups in C_s is a singleton group, and we get $|V_q| \leq 2s/q$. As C_s is 1-stable by the hypothesis,

there does not exist any couple of vertices $u, v \in V \setminus V_q$, such that $w(u, v) > 0$ whereas $C_s(u) \cap C_s(v) = \emptyset$ and, so, $m_{0,+} \leq 2sn/q \leq 2nf(C_s)/q$.

Finally, $\frac{f(C^+)}{f(C_s)} \leq 2h(q, w_p) \frac{m_{0,+}}{f(C_s)} = 2h(q, w_p) \frac{m_{1,+} + m_{0,+}}{f(C_s)} \leq 2h(q, w_p)(1 + 2\frac{n}{q})$. \square

5.2 Efficiency of higher order stability

Increasing k beyond 1 restricts further the set of stable configurations; they may even sometimes cease to exist. However, before this happens, we can expect the worst case configurations, and hence our general bounds, to improve. We prove in Theorem 12 that it is generally the case considering single channel case (the existence of a stable configuration is not guaranteed, but the result holds if it exists).

THEOREM 12. *Given \mathcal{W} , any graph $G = (V, E, w)$ with weights in \mathcal{W} satisfies, $f(C^+)/f(C_k^-) = O(\Delta_+)$ for $k \geq 2$, where $\Delta_+ = \max_{v \in V} |\{u \mid w(u, v) > 0\}|$.*

PROOF. Again, let m_+ be the number of edges $(u, v) \in E$ such that $w(u, v) > 0$. We have that $f(C^+) \leq 2m_+w_p$. Moreover, for every edge (u, v) of E such that $w(u, v) > 0$, for any k -stable configuration C_k for G , either $f_u(C_k) > 0$ or $f_v(C_k) > 0$. Indeed, suppose $f_u(C_k) = f_v(C_k) = 0$. Then $\{u, v\}$ breaks C_k by founding a new group, which is a contradiction. Thus, there is at least m_+/Δ_+ vertices v of V such that $f_v(C_k) > 0$, as the same node cannot be counted more than Δ_+ times. Consequently, $f(C^+)/f(C_k^-) = O(\Delta_+) = O(|V|)$. \square

In other words, we show that for $k \geq 2$ stable configurations are with a multiplicative factor of the optimal given by the maximum degree of a node. The result also holds for $k = 1$, for any \mathcal{W} such that neither $\{-M, 0, b\} \subseteq \mathcal{W}$ nor $\{-a, b\} \subseteq \mathcal{W}$. Indeed, in this case, if there exist u, v such that $f_u(C) = f_v(C) = 0$, whereas $w(u, v) > 0$, then u can reach the group of v , or vice-versa, and so, there is a 1-deviation.

We define the *price of k -anarchy* $p_a(n, k)$ as the greatest ratio $f(C^+)/f(C_k^-)$ obtained for a graph G of n nodes. By considering a worst case degree, we have $p_a(n, k) = O(n)$. For a given k , this bound is tight: we can even construct a graph with $\mathcal{W} \subseteq \mathbb{N}$ and any other non trivial weights such that $f(C^+)/f(C_k^-) = \Omega(n)$.

LEMMA 22. *Let k be any integer $k \geq 2$. Given integers $a, b, b', 0 < b' < b, a$, then there is graph G :*

- with $\mathcal{W} \supset \{-a, b\}$, $\frac{f(C^+)}{f(C_k^-)} \geq \frac{(k+1)\lfloor \frac{n}{(k+1)^2} \rfloor - 1}{k}$.
- with $\mathcal{W} \supset \{b, b'\}$, $\frac{f(C^+)}{f(C_k^-)} \geq 1 + \frac{b'}{b} \frac{kb(\lfloor \frac{n}{kb} \rfloor - 1)}{kb - 1}$.

PROOF. a) Case $\mathcal{W} \supset \{-a, b\}$

Let $n' = (k+1)^2 \lfloor \frac{n}{(k+1)^2} \rfloor \leq n$. We build a network $G = (V, E, w)$ such that $|V| = n'$, as follows. The vertices in V are labeled $v_{i,j}$, with $1 \leq i \leq k+1$, and with $1 \leq j \leq n'/(k+1)$. For every $1 \leq i \leq k+1$, the row L_i is the set $\{v_{i,1}, v_{i,2}, \dots, v_{i, \frac{n'}{k+1}}\}$. In the same way, for every $1 \leq j \leq \frac{n'}{k+1}$, the column C_j is the set $\{v_{1,j}, \dots, v_{k+1,j}\}$. Furthermore, vertices $v_{i,j}$ and $v_{l,t}$ are friends, that is $w(v_{i,j}, v_{l,t}) = b > 0$, if, and only if, either $i = l$ or $j = t$. Otherwise,

$$w(v_{i,j}, v_{l,t}) = -a \leq 0.$$

Let $C_a = (L_1, L_2, \dots, L_{k+1})$ be a configuration for G . We have $f(C_a) = n'b(\frac{n'}{k+1} - 1)$. On the other hand, we present $C_b = (C_1, C_2, \dots, C_{\frac{n'}{k+1}})$, and we claim that C_b is a k -stable configuration for G . Indeed, observe that for any $1 \leq j \leq \frac{n'}{k+1}$, any vertex $v_{i,j} \in C_j$ has one, and only one friend, in any other column. Hence, if $v_{i,j}$ were part of a subset S that breaks C_b , then after the k -deviation, $v_{i,j}$ would have, at most, k friends in its group, which is already the case in C_j . As a consequence, there is no vertex that can strictly increase its utility by a k -deviation, and so, the claim is proved. However, $f(C_b) = n'bk$. Finally, $\frac{f(C^+)}{f(C_k^-)} \geq \frac{f(C_a)}{f(C_b)} = \frac{\frac{n'}{k+1} - 1}{k} = \frac{(k+1)\lfloor \frac{n}{(k+1)^2} \rfloor - 1}{k}$.

b) Case $\mathcal{W} \supset \{b, b'\}$

Let $n' = kb \lfloor \frac{n}{kb} \rfloor$. We build the network $G = (V, E, w)$ with \mathcal{W} , such that $|V| = n'$ as follows. We partition the set of vertices V into $\frac{n'}{kb}$ parts, denoted $A_1, A_2, \dots, A_{\frac{n'}{kb}}$, of equal size kb . For every couple of vertices u, v , $w(u, v) = b$ if u and v are in the same part A_i , with $1 \leq i \leq \frac{n'}{kb}$, and $w(u, v) = b'$ otherwise.

Obviously, a maximum (stable) configuration for G is $C_a = (V)$, and $f(C_a) = n'[b(kb - 1) + b'(n' - kb)]$.

Furthermore, we claim that $C_b = (A_1, A_2, \dots, A_{\frac{n'}{kb}})$ is a k -stable configuration too. Indeed, any vertex v that might break the configuration C_b , by joining a coalition S whose size is greater or equal to k , would have its individual utility after the deviation that is at most $(k-1)b + kbb' = kb(1 + b') - b \leq kb^2 - b = (kb - 1)b$. Hence C_b is k -stable, and $f(C_b) = n'b(kb - 1)$.

Finally, $\frac{f(C_a)}{f(C_b)} = 1 + \frac{b'}{b} \frac{n' - kb}{kb - 1} = 1 + \frac{b'}{b} \frac{kb(\lfloor \frac{n}{kb} \rfloor - 1)}{kb - 1}$. \square

6. REFERENCES

- [1] L. Blume, D. Easley, J. M. Kleinberg, R. Kleinberg, and É. Tardos. Network formation in the presence of contagious risk. In *Proceedings of ACM EC*, 2011.
- [2] T. Brylawski. The lattice of integer partitions. *Discrete Mathematics*, 6(3):201 – 219, 1973.
- [3] D. P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are np-complete. *Discrete Mathematics*, 30(3):289 – 293, 1980.
- [4] A. Galeotti and S. Goyal. The law of the few. *American Economic Review*, 100(4):1468–1492, 2010.
- [5] E. Goles and M. A. Kiwi. Games on line graphs and sand piles. *Theoretical Computer Science*, 115(2):321 – 349, 1993.
- [6] M. Jackson. A survey of models of network formation: Stability and efficiency. *Group Formation in Economics: Networks, Clubs and Coalitions*, 2003.
- [7] S. Kairam, M. Brzozowski, D. Huffaker, and E. Chi. Talking in circles: selective sharing in google+. In *Proceedings of ACM CHI*, May 2012.
- [8] R. Karp. Reducibility among combinatorial problems. In R. Miller and J. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. 1972.
- [9] J. M. Kleinberg and K. Ligett. Information-Sharing and Privacy in Social Networks. *paper in progress*

(available at arxiv.org/abs/1003.0469), 2010.

- [10] R. Korf. Multi-way number partitioning. *International Joint Conference on Artificial Intelligence*, 2009.
- [11] F. Ming, F. Wong, and P. Marbach. “Who Are Your Friends?”—A Simple Mechanism that Achieves Perfect Network Formation. *INFOCOM, 2011 Proceedings IEEE*, pages 566–570, 2011.
- [12] D. Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(1):103–128, 2007.